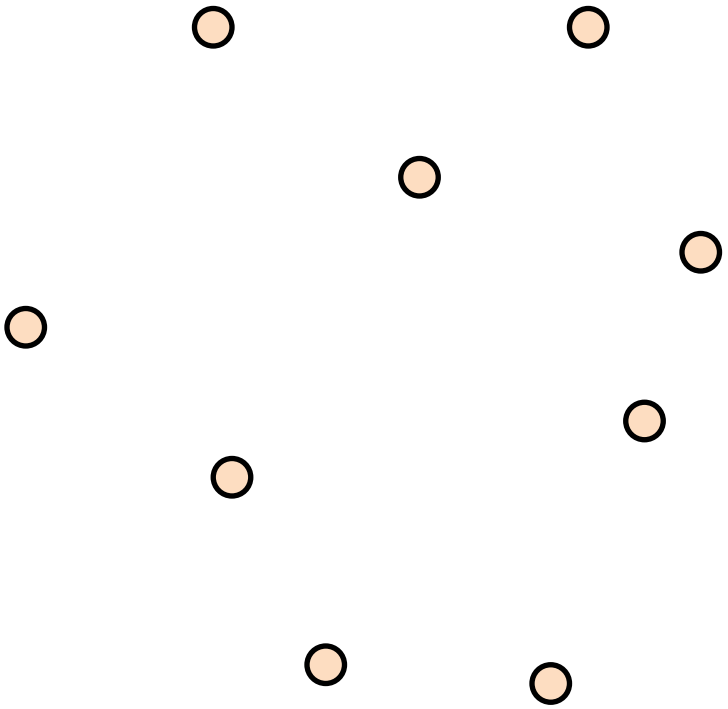
2.7 Geometrische Algorithmen

2.7 Geometrische Algorithmen

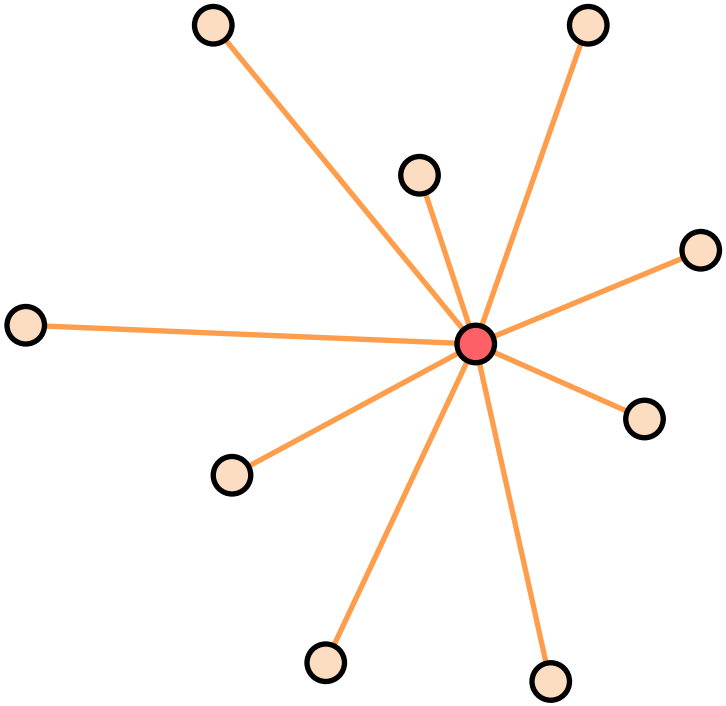
Generierung von Polygonen

- Gegeben sei eine ungeordnete Menge von Eckpunkten p_i
- Generiere ein einfaches, geschlossenes Polygon (durch “Sortieren” der p_i)

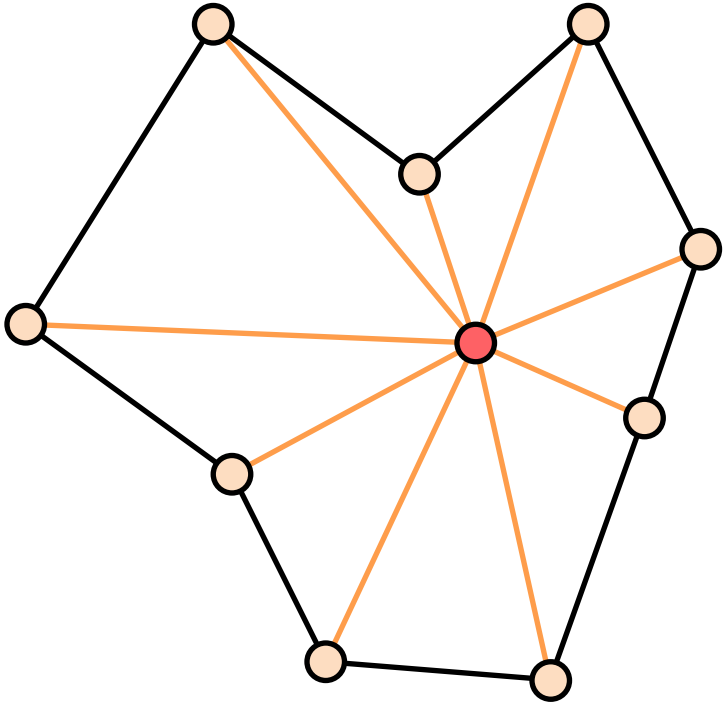
Generierung von Polygonen



Generierung von Polygonen

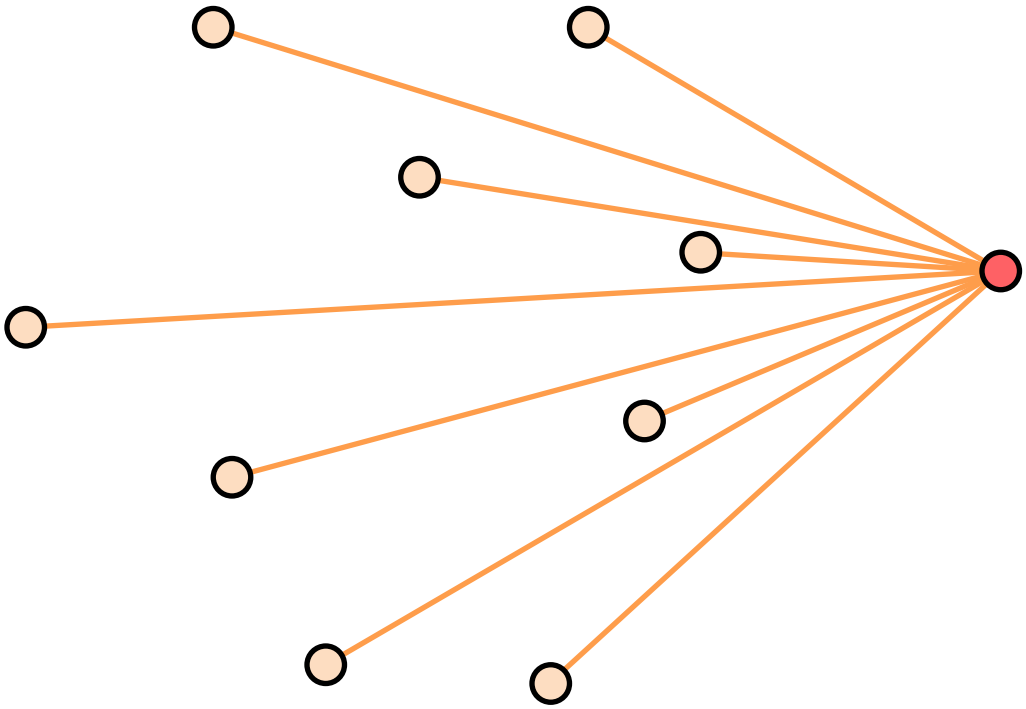


Generierung von Polygonen



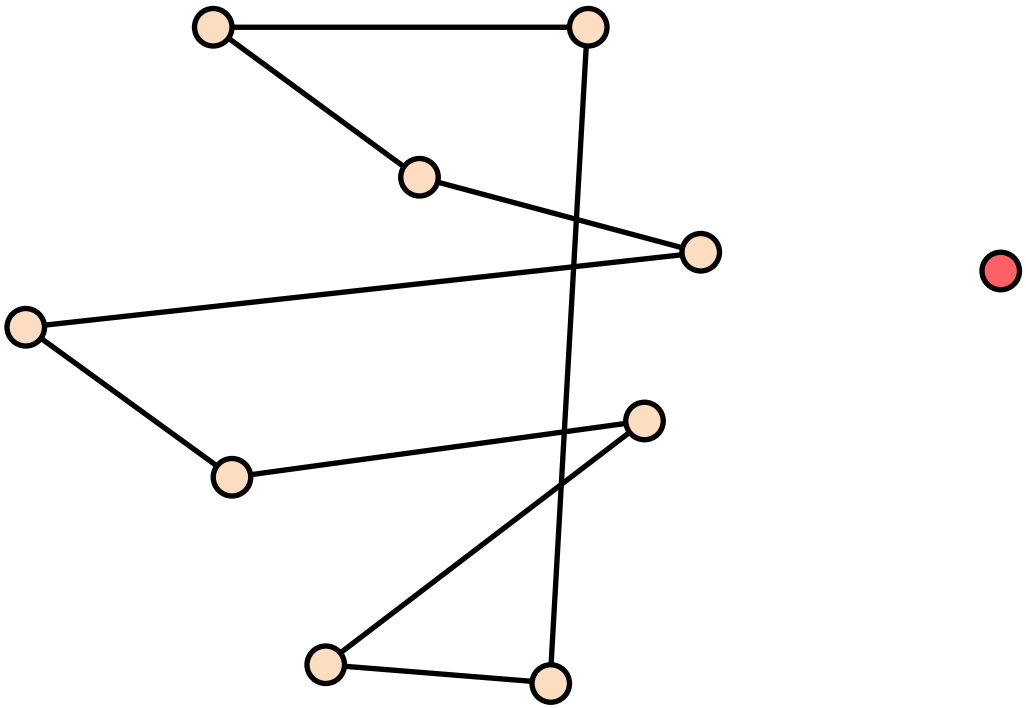
Generierung von Polygonen

aber...



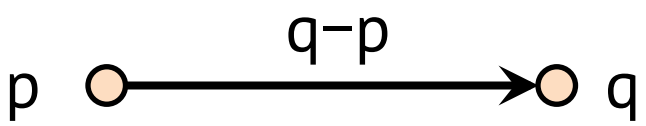
Generierung von Polygonen

aber...



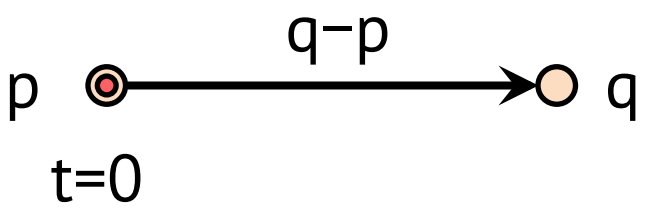
Konvexität

- Eine Teilmenge S des R^2 heißt konvex, wenn mit jedem Paar p,q von Punkten aus S auch alle Zwischenpunkte $(1-t)p+tq$ mit $0 \leqslant t \leqslant 1$ in S liegen.
- $(1-t)p+tq = p-tp+tq = p+t(q-p)$



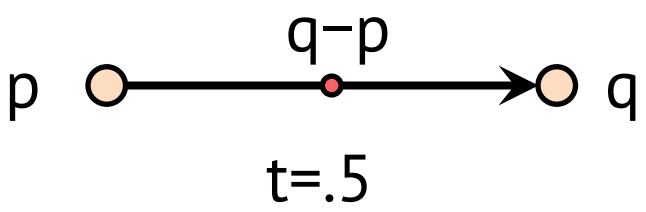
Konvexität

- Eine Teilmenge S des R^2 heißt konvex, wenn mit jedem Paar p,q von Punkten aus S auch alle Zwischenpunkte (1−t)p +tq mit $0 \leqslant t \leqslant 1$ in S liegen.
- (1−t)p+tq = p−tp+tq = p+t(q−p)



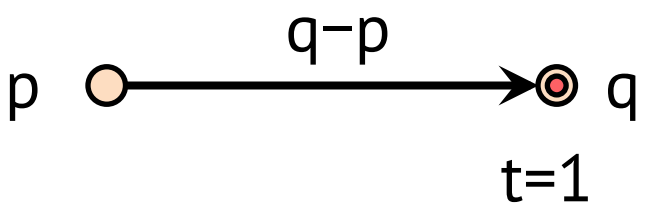
Konvexität

- Eine Teilmenge S des R^2 heißt konvex, wenn mit jedem Paar p,q von Punkten aus S auch alle Zwischenpunkte $(1-t)p+tq$ mit $0 \leqslant t \leqslant 1$ in S liegen.
- $(1-t)p+tq = p-tp+tq = p+t(q-p)$

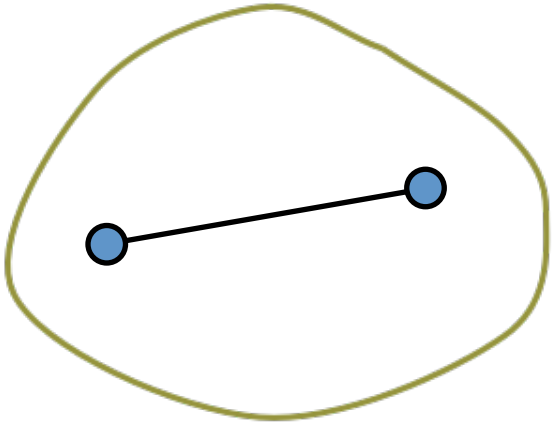


Konvexität

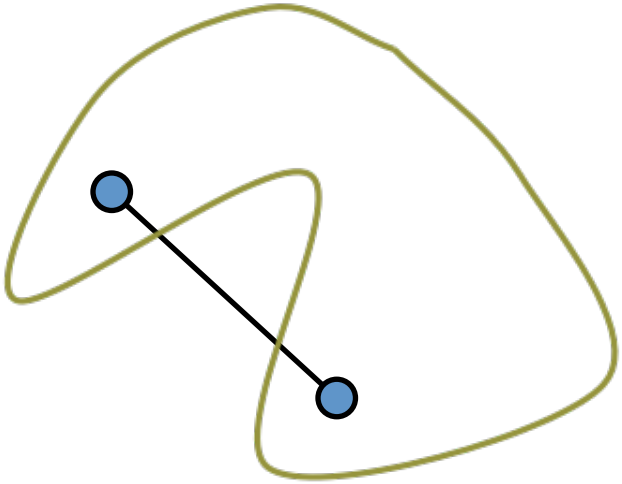
- Eine Teilmenge S des R^2 heißt konvex, wenn mit jedem Paar p,q von Punkten aus S auch alle Zwischenpunkte $(1-t)p+tq$ mit $0 \leqslant t \leqslant 1$ in S liegen.
- $(1-t)p+tq = p-tp+tq = p+t(q-p)$



Konvexität



konvex



nicht konvex

Konvexe Hülle

- Definition
  Die Konvexe Hülle CH(S) einer Teilmenge S des R^2 ist die kleinste konvexe Menge (bzgl. $\subseteq$), die S enthält.
- Lemma

$$CH(S) = \bigcap_{K \supseteq S,\ K \text{ ist konvex}} K$$

Konvexe Hülle

- Lemma
  Es sei eine Menge von (Eck-)Punkten p_i gegeben. Die Konvexe Hülle $CH(p_i)$ ist die Menge aller Punkte q mit

$$q = \sum_{i=1}^{n} \alpha_i p_i \text{ mit } \sum_{i=1}^{n} \alpha_i = 1 \text{ und } \alpha_i \geq 0$$

Konvexe Hülle

- Beweis

  Es sei $C = \left\{ \sum_{i=1}^{n} \alpha_i p_i : \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0 \right\}$

  Zu zeigen: C = CH(p_i)

- Wegen

$$\sum_{i=1}^{n} \alpha_i p_i = \alpha_1 p_1 + (1 - \alpha_1) \sum_{i=2}^{n} \frac{\alpha_i}{1 - \alpha_1} p_i$$

  gilt offensichtlich $C \subseteq CH(p_i)$

- Noch zu zeigen: C ist konvex

Konvexe Hülle

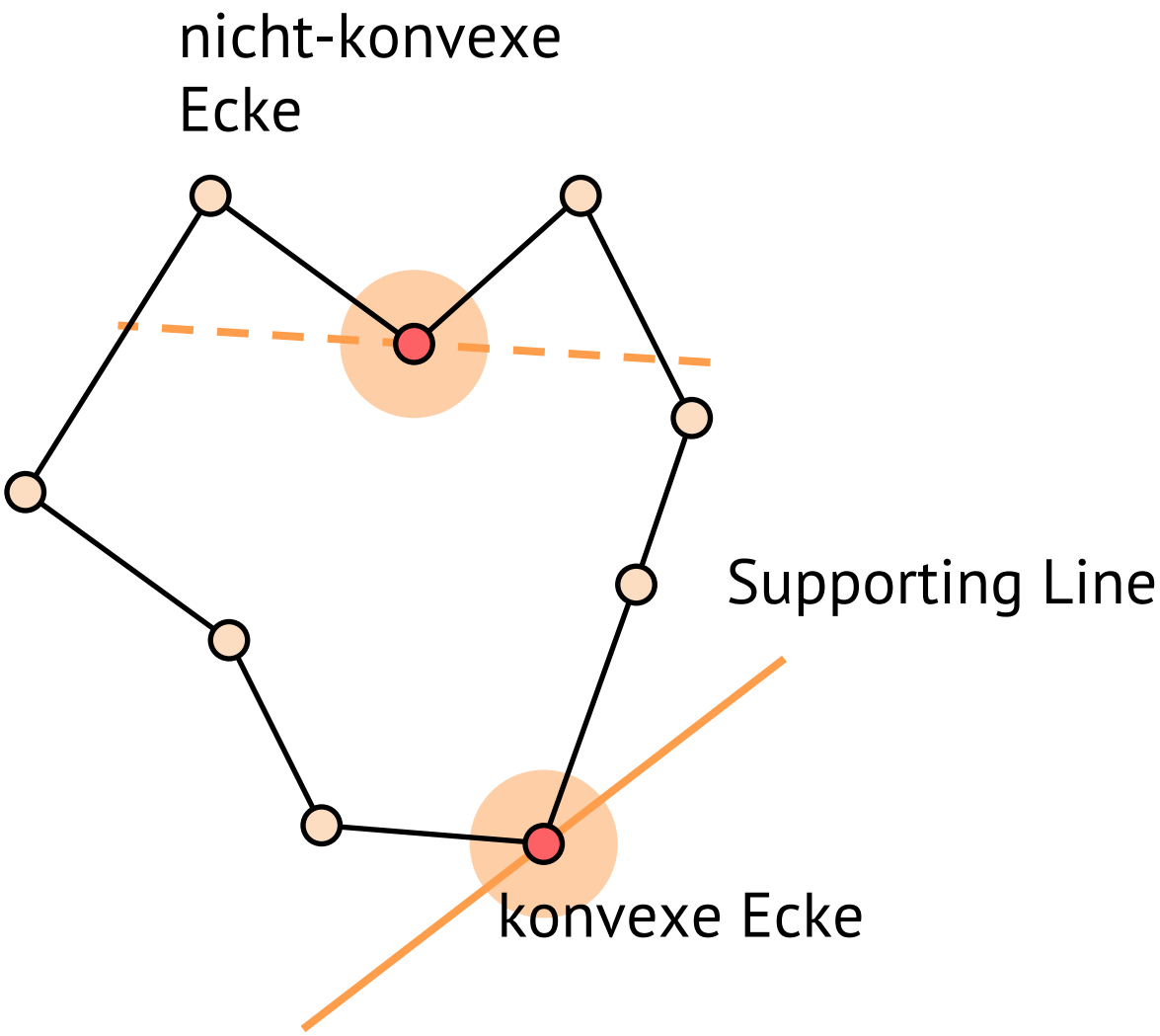
$$q = \sum_{i=1}^{n} \alpha_i p_i, \quad \sum_{i=1}^{n} \alpha_i = 1, \quad \alpha_i \geq 0$$
$$r = \sum_{i=1}^{n} \beta_i p_i, \quad \sum_{i=1}^{n} \beta_i = 1, \quad \beta_i \geq 0$$

$$\begin{aligned}(1-t)q + tr &= (1-t)\sum_{i=1}^{n} \alpha_i p_i + t\sum_{i=1}^{n} \beta_i p_i \\ &= \sum_{i=1}^{n} (1-t)\alpha_i p_i + t\beta_i p_i \\ &= \sum_{i=1}^{n} ((1-t)\alpha_i + t\beta_i) p_i\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} ((1-t)\alpha_i + t\beta_i) &= (1-t)\sum_{i=1}^{n} \alpha_i + t\sum_{i=1}^{n} \beta_i \\ &= (1-t) + t \\ &= 1\end{aligned}$$

$$(1-t)\alpha_i + t\beta_i \geq 0$$

Lokale Konvexität

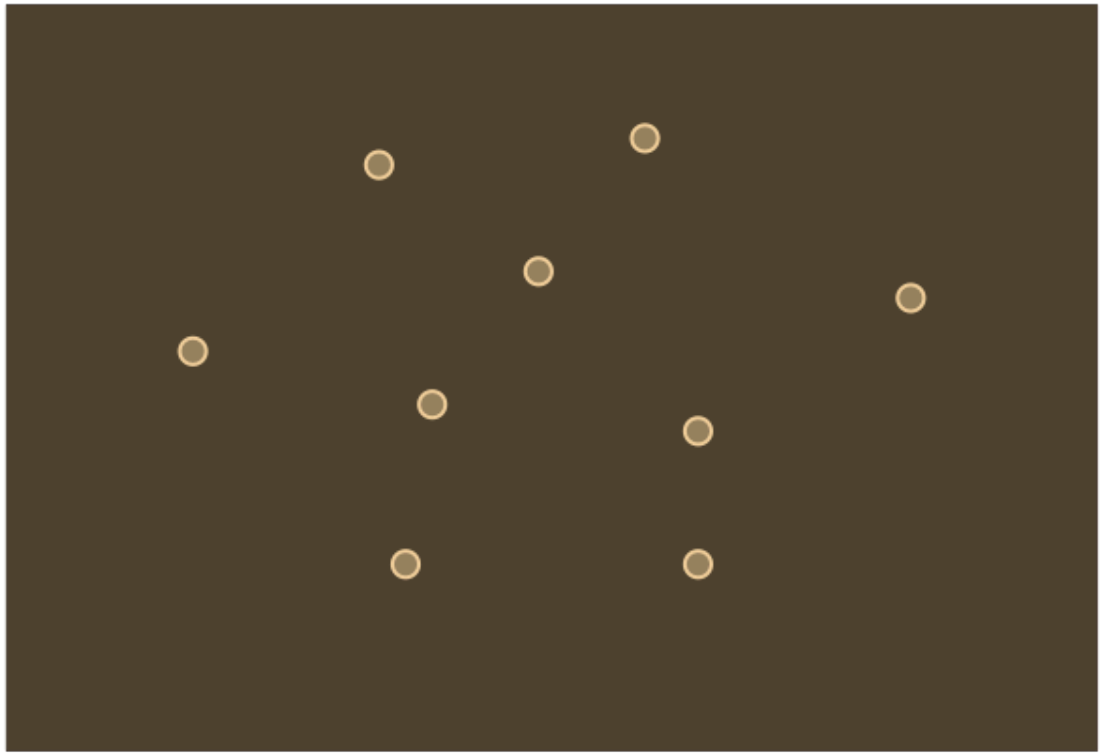


Konvexe Hülle

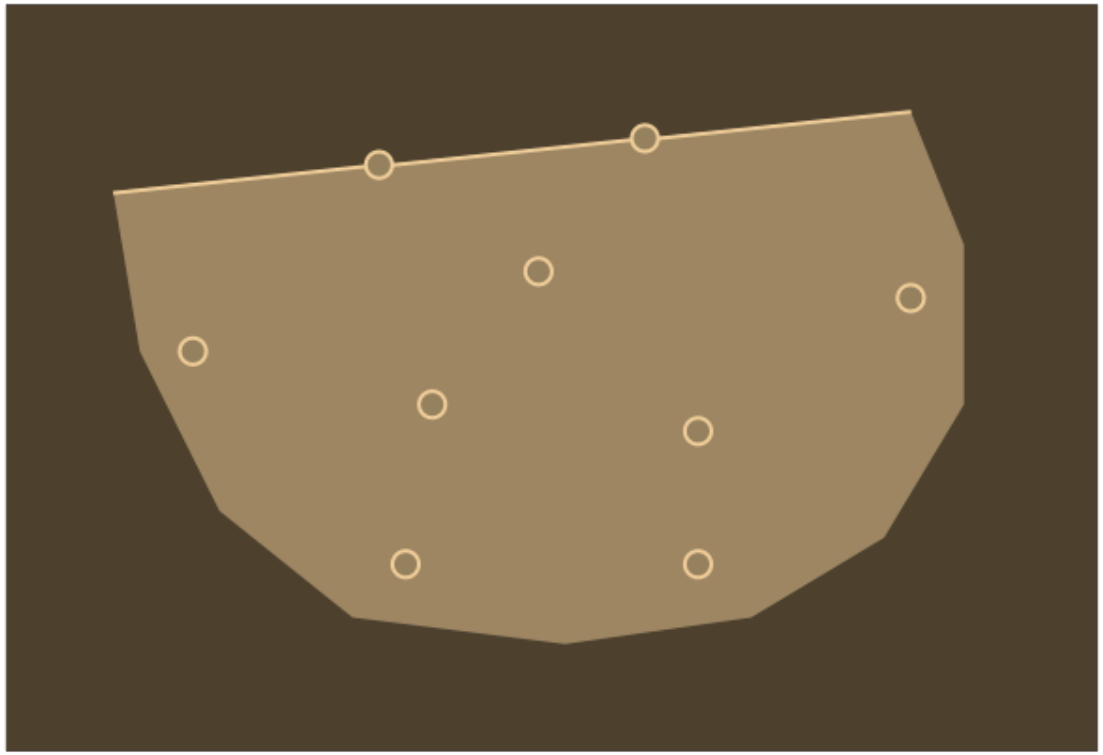
- Jede Supporting Line definiert einen Halbraum, in dem sich alle Eckpunkte befinden.

- Die konvexe Hülle ergibt sich aus dem Schnitt aller dieser Halbräume (ohne Beweis).

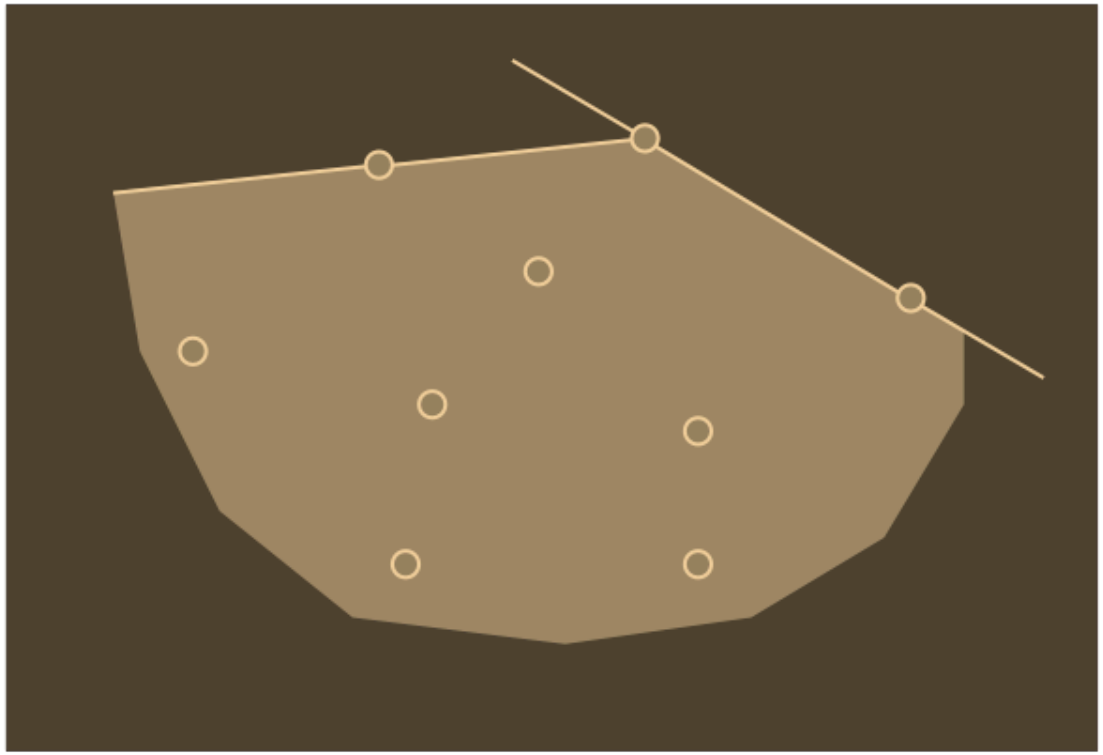
Konvexe Hülle



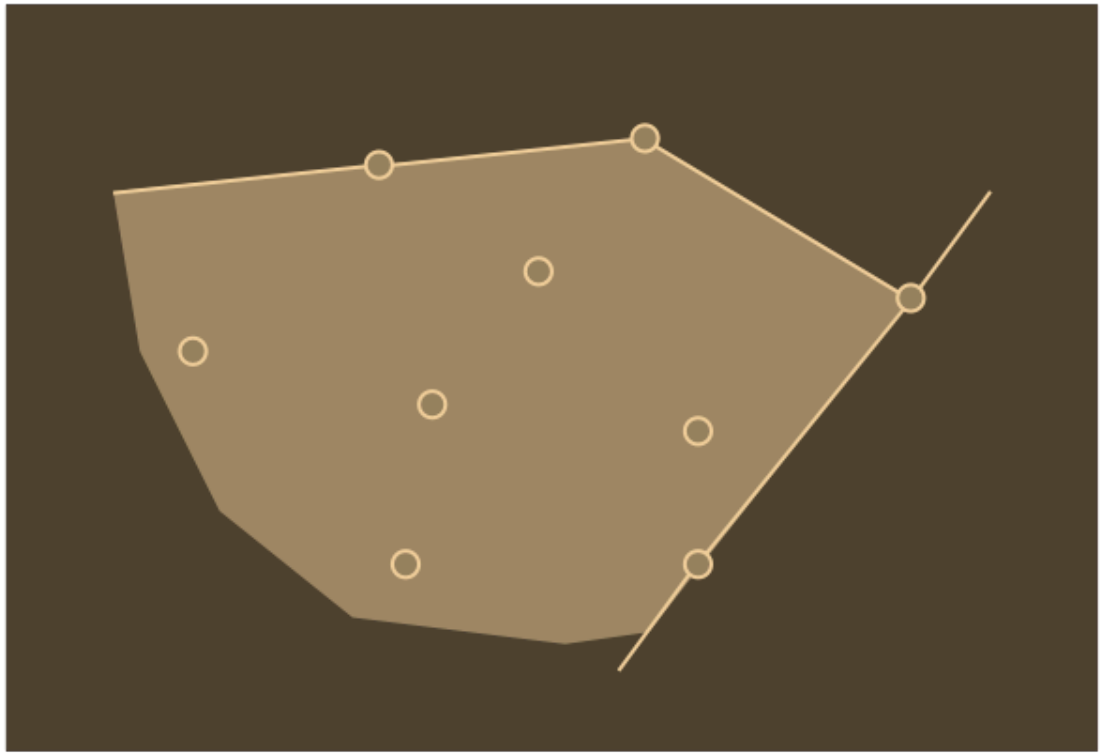
Konvexe Hülle



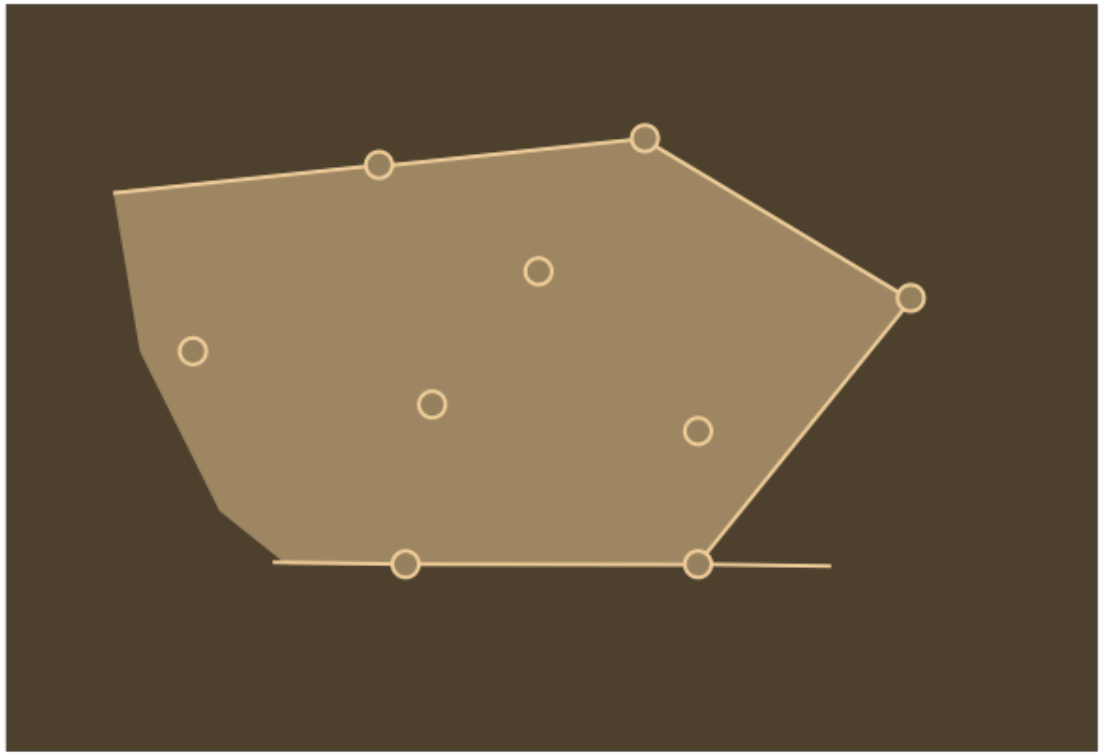
Konvexe Hülle



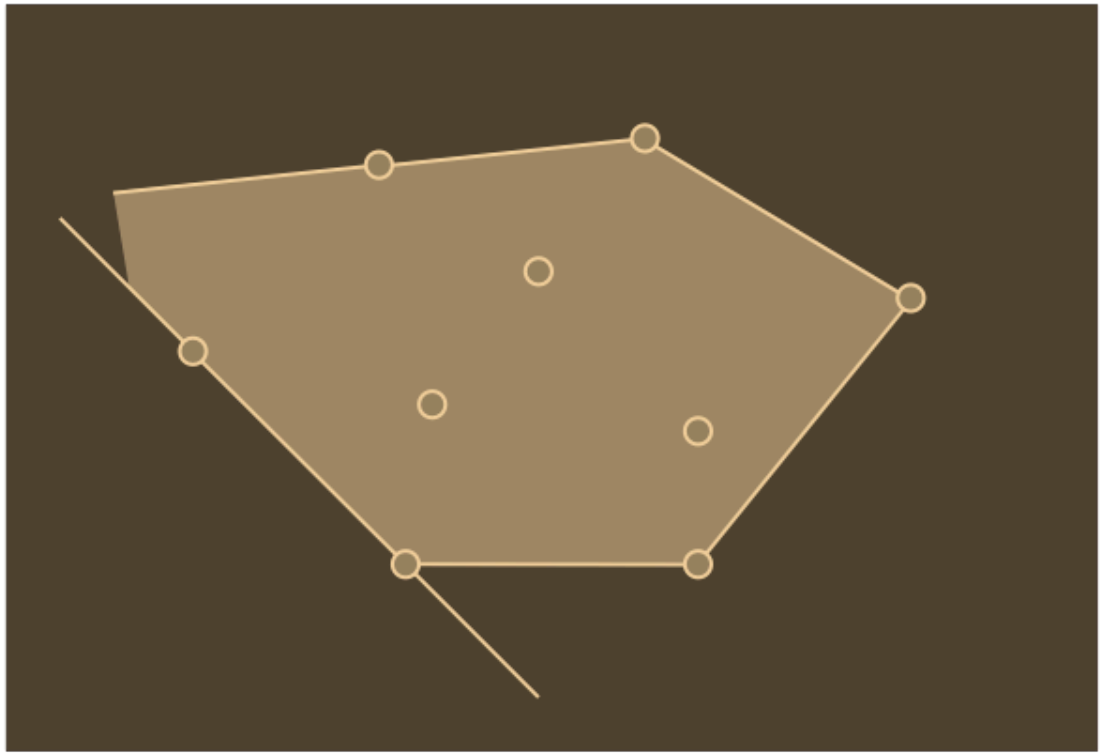
Konvexe Hülle



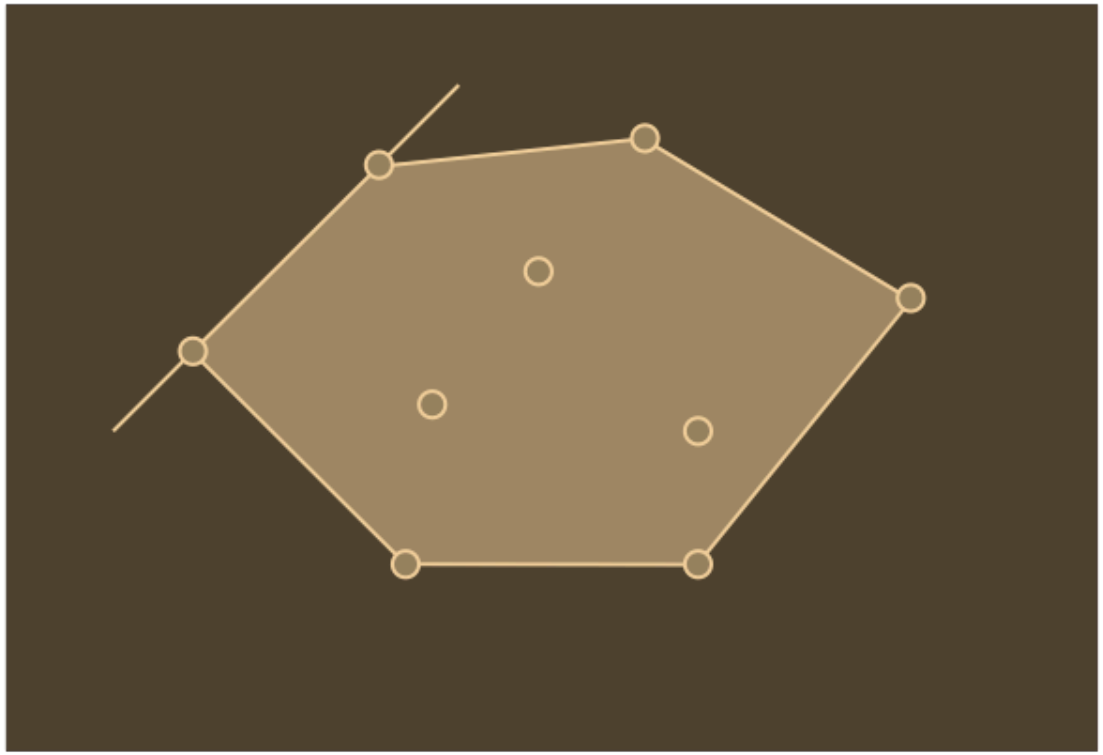
Konvexe Hülle



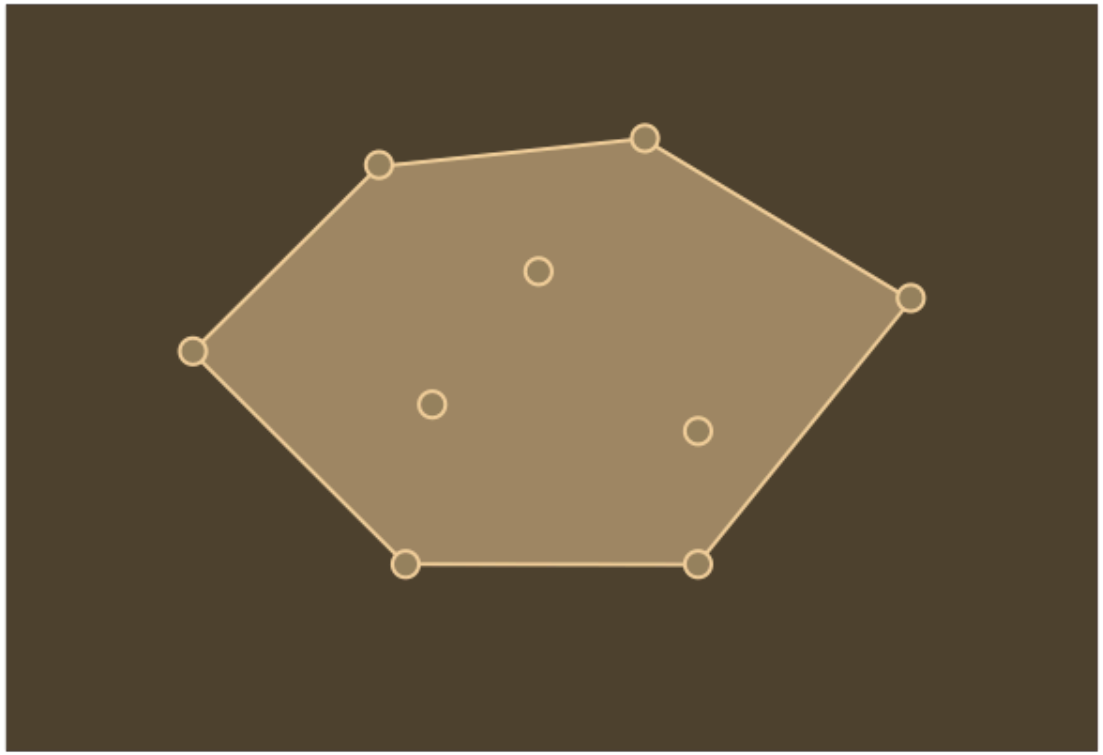
Konvexe Hülle



Konvexe Hülle



Konvexe Hülle



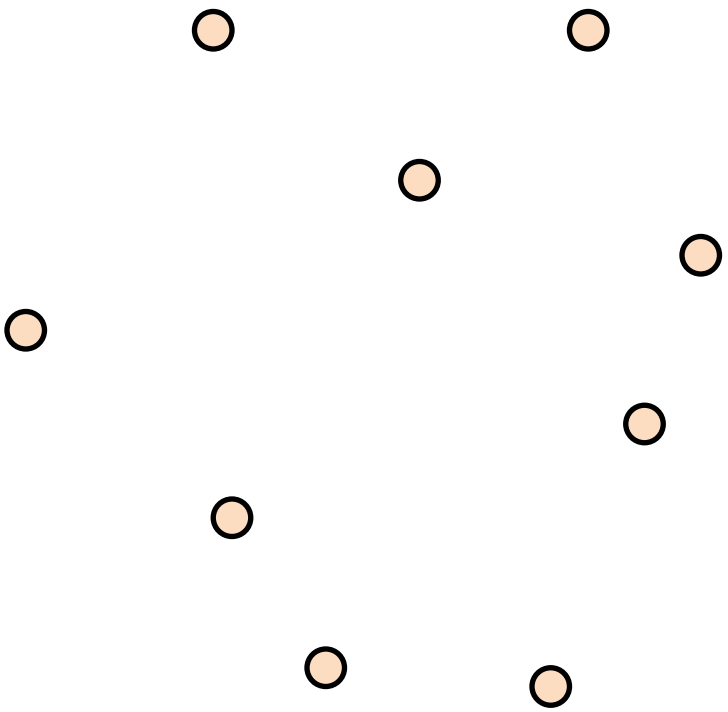
Lokale Konvexität

- Lemma
  Der Eckpunkt mit der kleinsten x-Koordinate (und ggfs. mit der kleinsten y-Koordinate) ist immer konvex.
- Beweis: Supporting Line parallel zur y-Achse

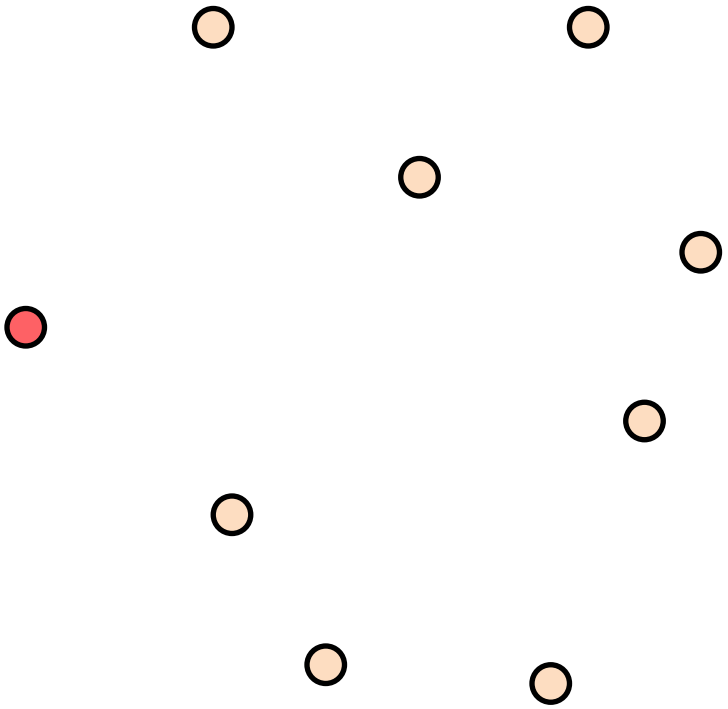
Generierung von Polygonen

- Suche eine konvexe Ecke
- Sortiere die übrigen Ecken nach dem Winkel zur x-Achse
- Verbinde aufeinanderfolgende Ecken

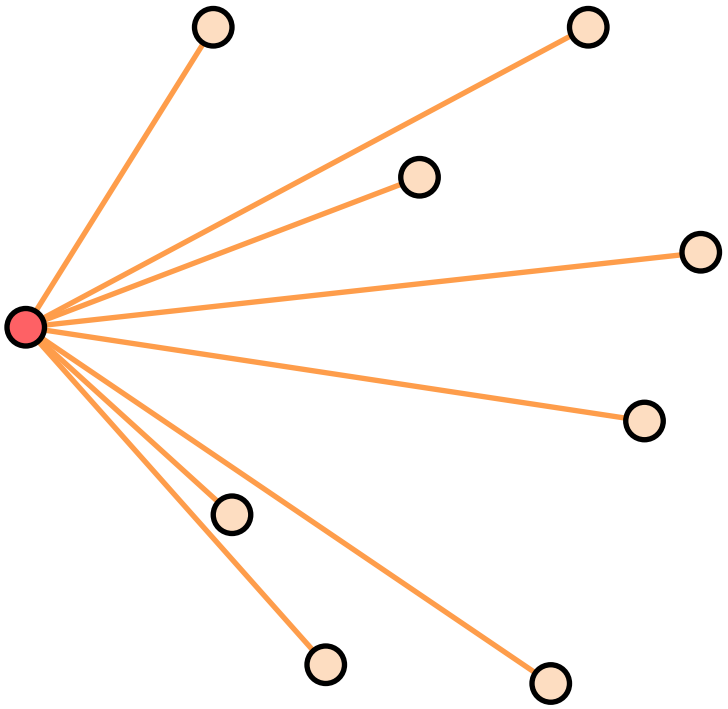
Generierung von Polygonen



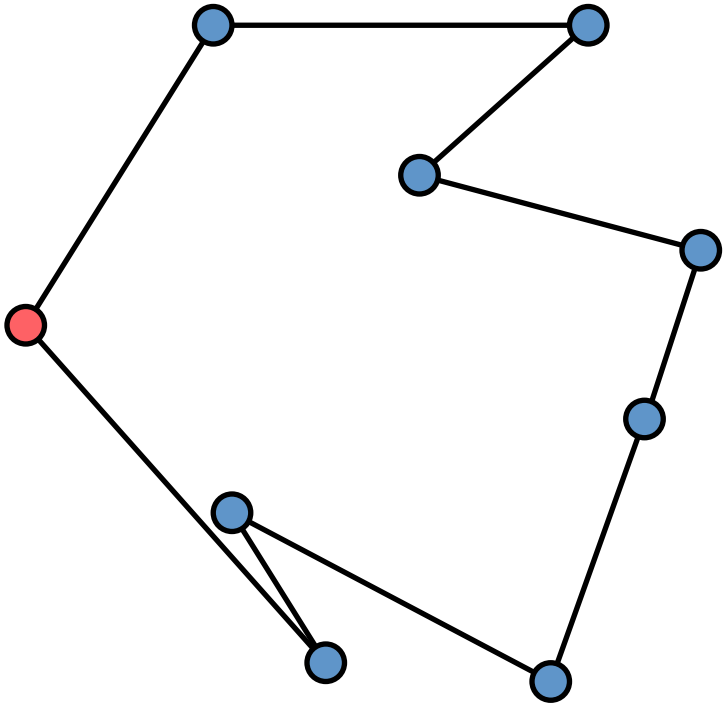
Generierung von Polygonen



Generierung von Polygonen



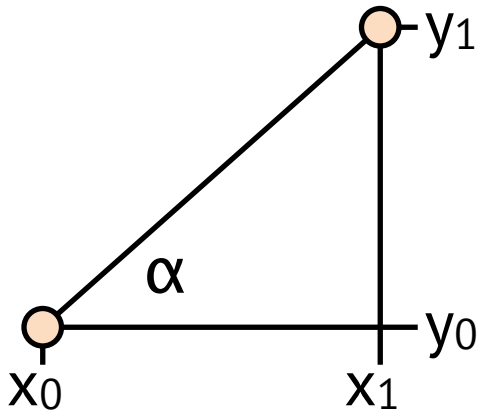
Generierung von Polygonen



Sortierung nach Winkel

- Winkel der Kante von (x_0,y_0) nach (x_1,y_1) zur x-Achse

$$\tan \alpha = \frac{y_1 - y_0}{x_1 - x_0}$$

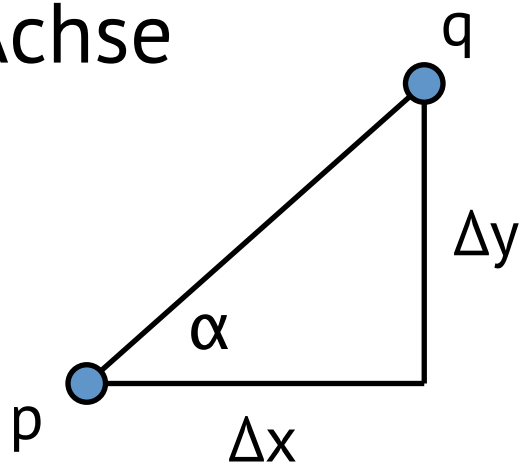


$$(\sin \alpha)^2 = \frac{(y_1 - y_0)^2}{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

Sortierung nach Winkel

- Winkel $\alpha(p,q)$ der Kante von p=(x,y) nach q=(x+Δx,y+Δy) zur x-Achse

$$\tan \alpha = \frac{\Delta y}{\Delta x}$$



- Nachteile
  - Berechnung des Arcustangens ist teuer.
  - Abfangen von Division durch 0.
  - Prüfung, in welchem Quadrant der Punkt liegt notwendig.

Sortierung nach Winkel

- Besser: Wähle einfache Funktion β mit
  $\beta(p,q_0) < \beta(p,q_1) \Leftrightarrow \alpha(p,q_0) < \alpha(p,q_1)$
- Wähle z.B. im ersten Quadrant

$$\beta(p,q) = \frac{\Delta y}{\Delta x + \Delta y} \in [0,1]$$

- Beta(dx,dy)

```
beta ← dy / (abs(dx) + abs(dy))
if dx < 0 then return 2 − beta
if dy < 0 then return 4 + beta
return beta
```

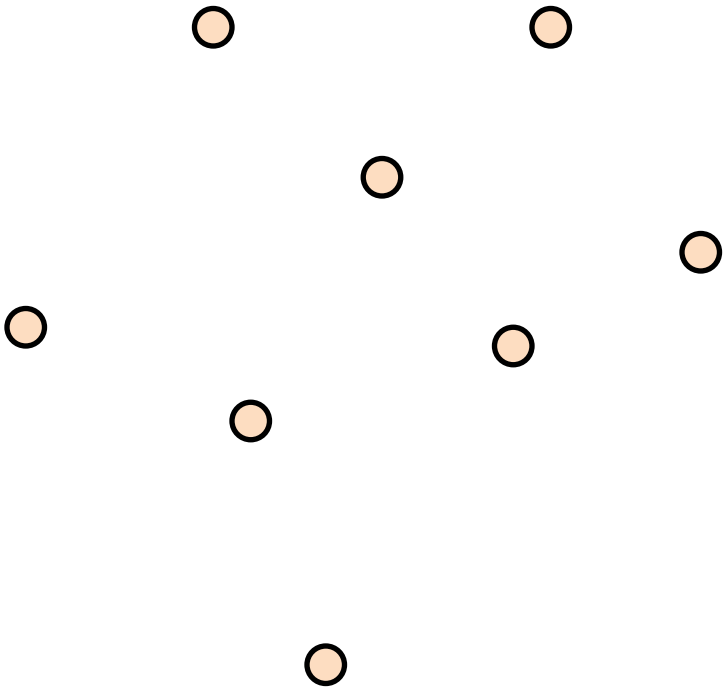

Konvexe Hülle

- Wir nehmen immer an, dass die Punkte in der konvexen Hülle einer Punktmenge so sortiert sind, dass das zugehörige Polygon gegen den Uhrzeigersinn orientiert ist.

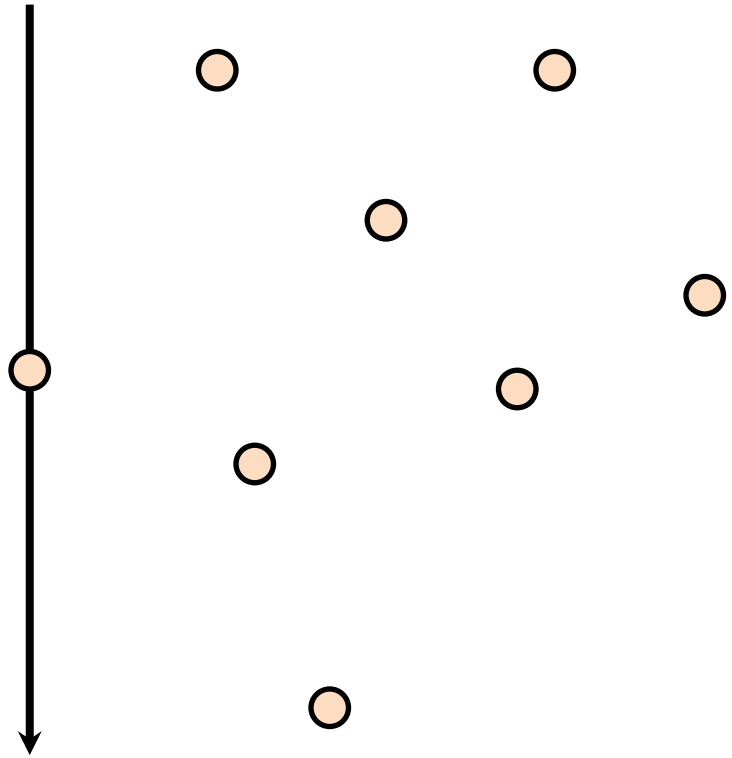
Gift Wrapping

- Beginne mit einer Supporting Line.
- Ergänze in jedem Schritt den Eckpunkt, der den geringsten Winkel zur aktuellen Supporting Line aufspannt.

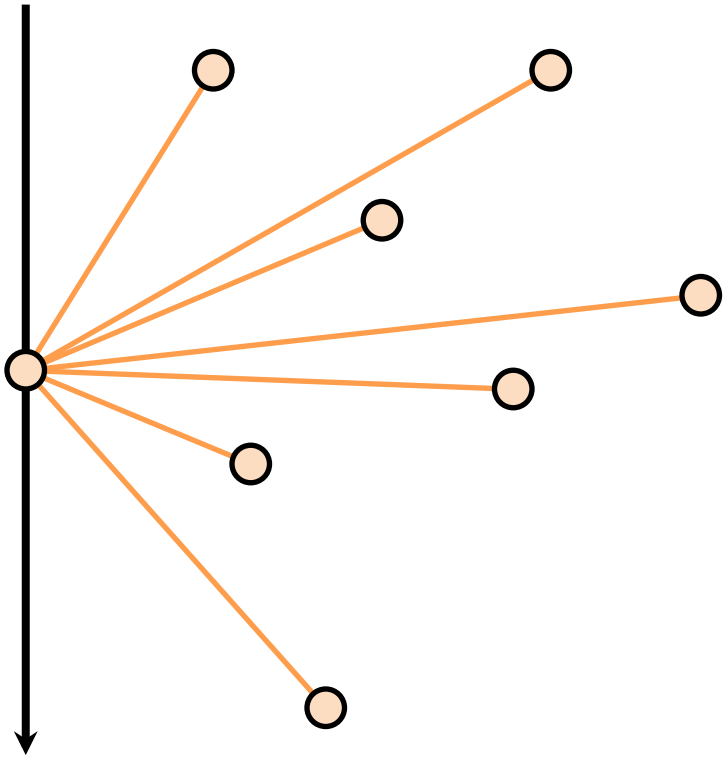
Gift Wrapping



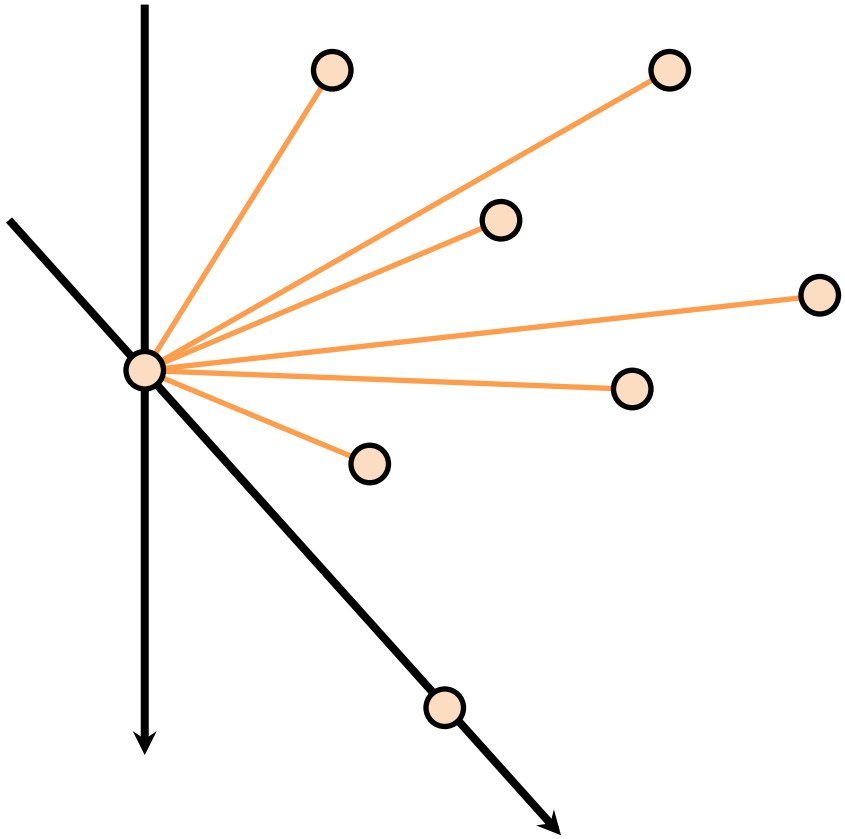
Gift Wrapping



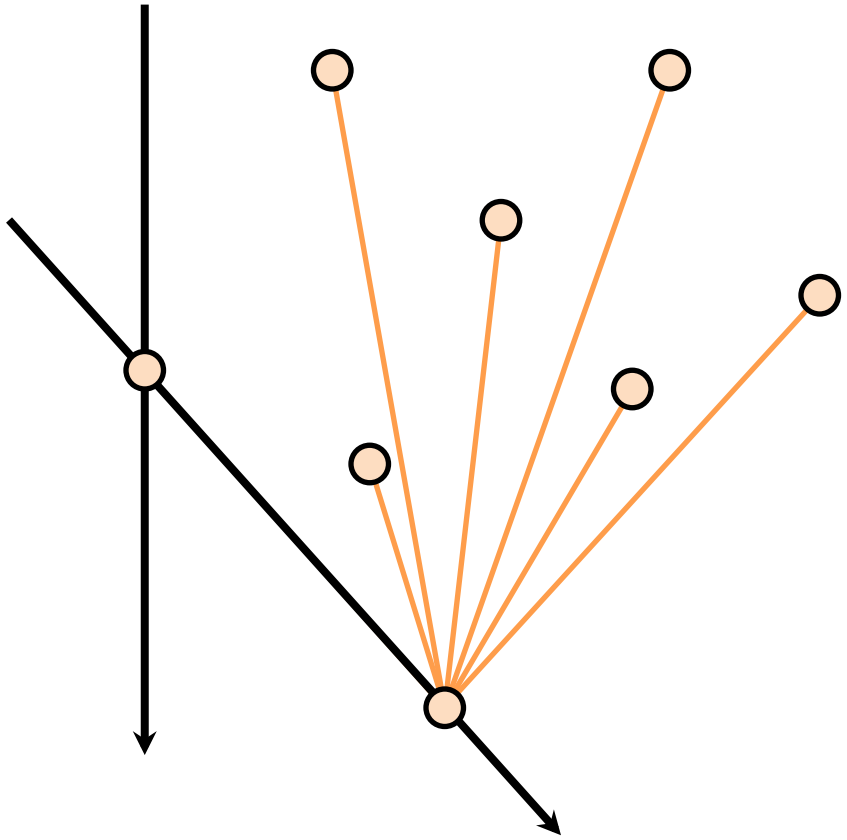
Gift Wrapping



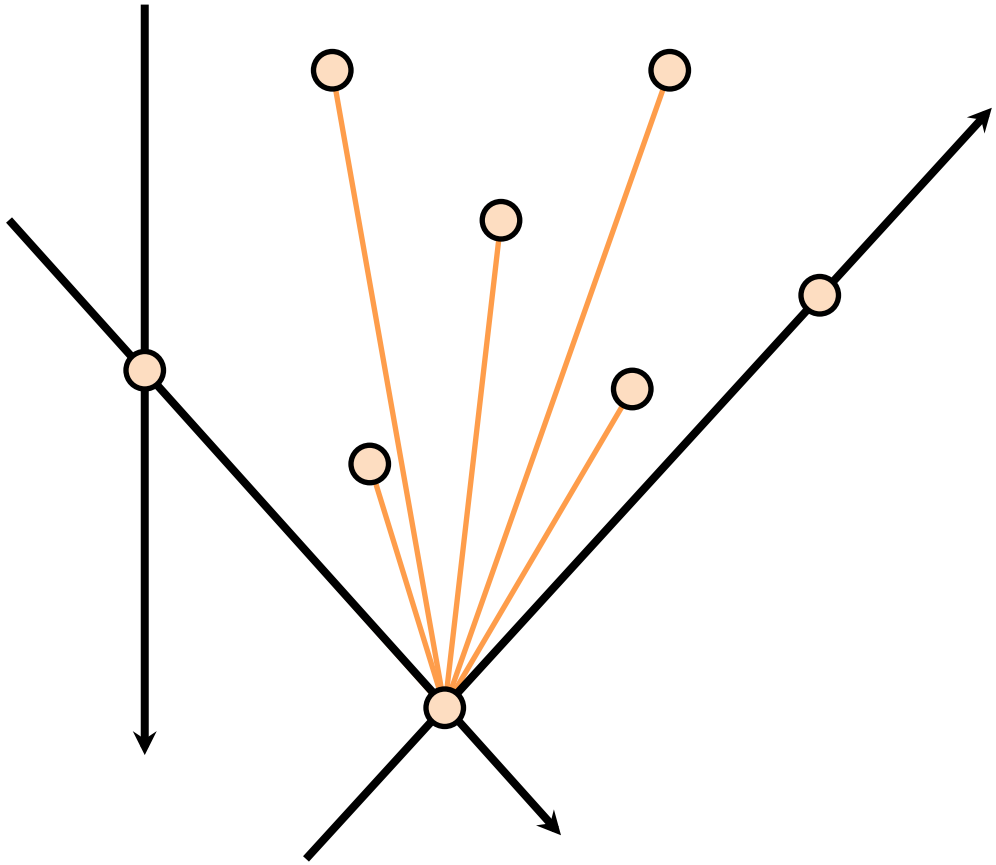
Gift Wrapping



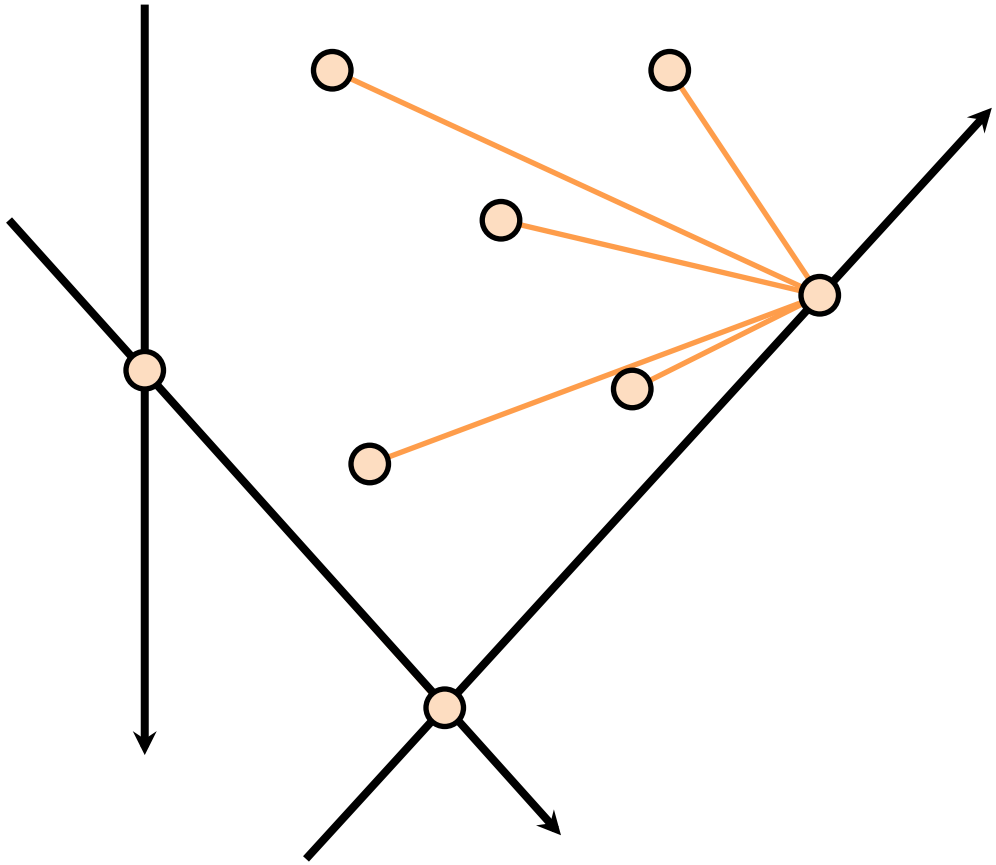
Gift Wrapping



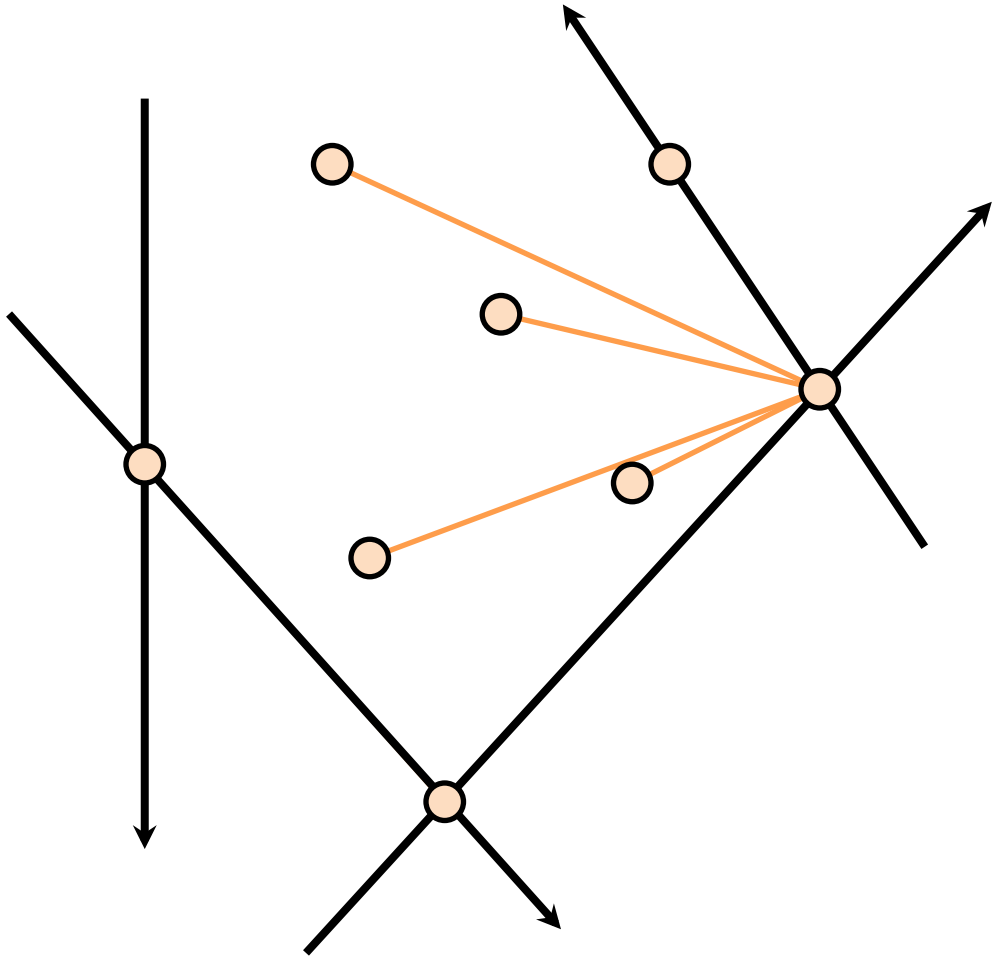
Gift Wrapping



Gift Wrapping



Gift Wrapping



Gift Wrapping

- GiftWrap(pts[1..n])
  - p ← LeftMostPoint(pts)
  - q ← Null
  - CH.create()
  - CH.add(p)
  - L ← negative y-axis
  - while q ≠ CH.top() do
    - q ← select point with smallest angle (p,L)
    - CH.add(q)
    - L ← Line(p,q)
    - p ← q

Gift Wrapping

- Die Gift Wrapping Prozedur entspricht im Wesentlichen dem Selection Sort.
- Aufwand $O(m \times n)$, wobei m die Zahl der Eckpunkte in der konvexen Hülle ist
- Best case: $m = 3$
- Worst case: $m = n$

2.7 Geometrische Algorithmen

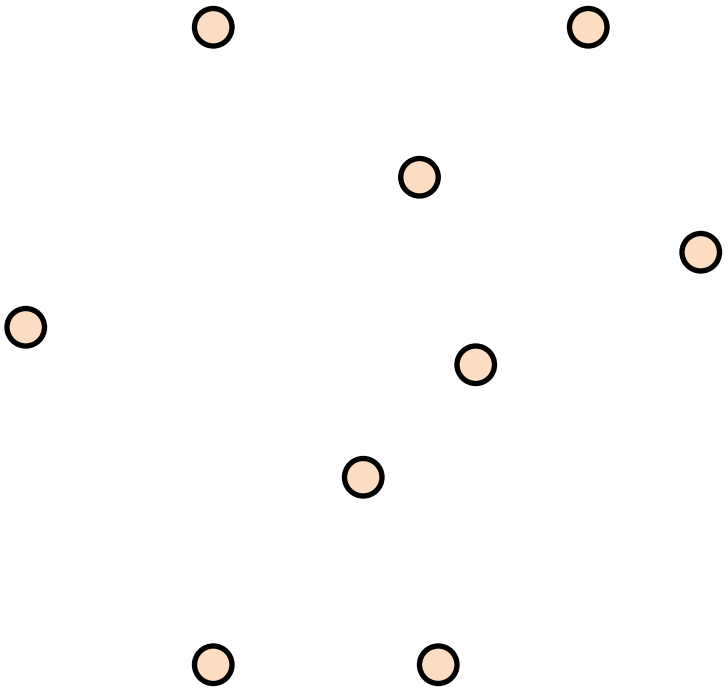
Graham's Scan

- Gift-Wrapping führt zuviele redundante Berechnungen durch.
- Bei der einfachen Generierung von Polygonen genügt ein einziger globaler Sortierungsschritt.
- Idee: Entferne die nicht-konvexen Ecken aus dem einfachen Polygon.

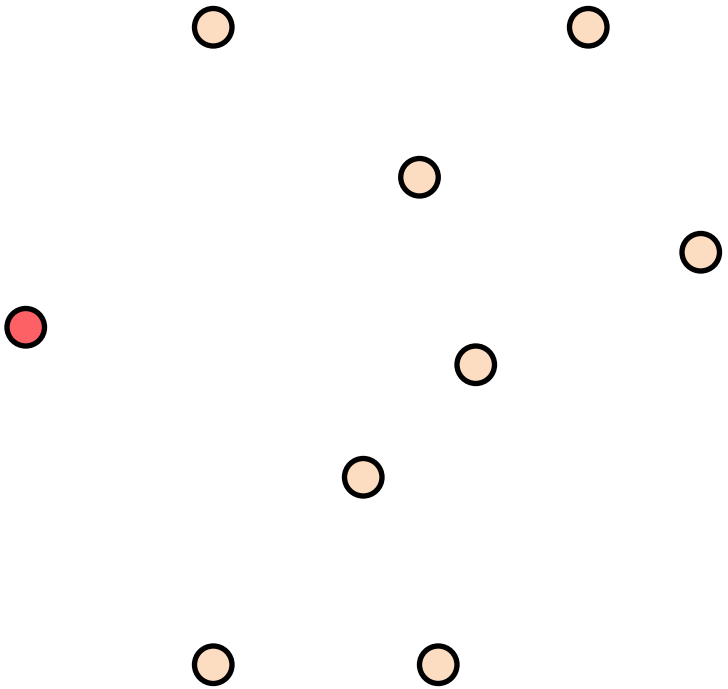
Graham's Scan

- GrahamScan(pts)
  let p be the left-most point in pts
  sort pts by angle of line (p,pts[i]) to x-axis
  p ← head(pts); q ← next(p); r ← next(q)
  while r ≠ p do
    if p,q,r is convex do
      p ← next(p); q ← next(q); r ← next(r)
    else
      delete q
      q ← p; p ← prev(p)

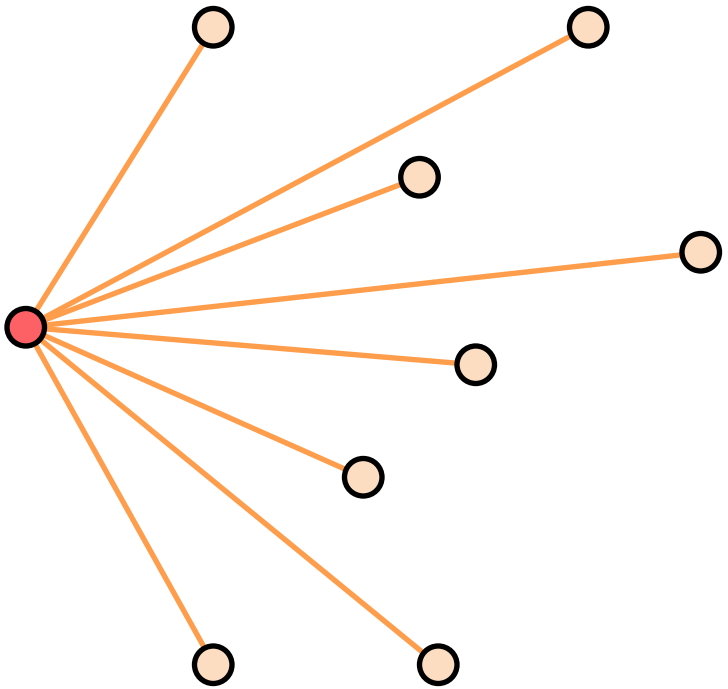
Graham's Scan



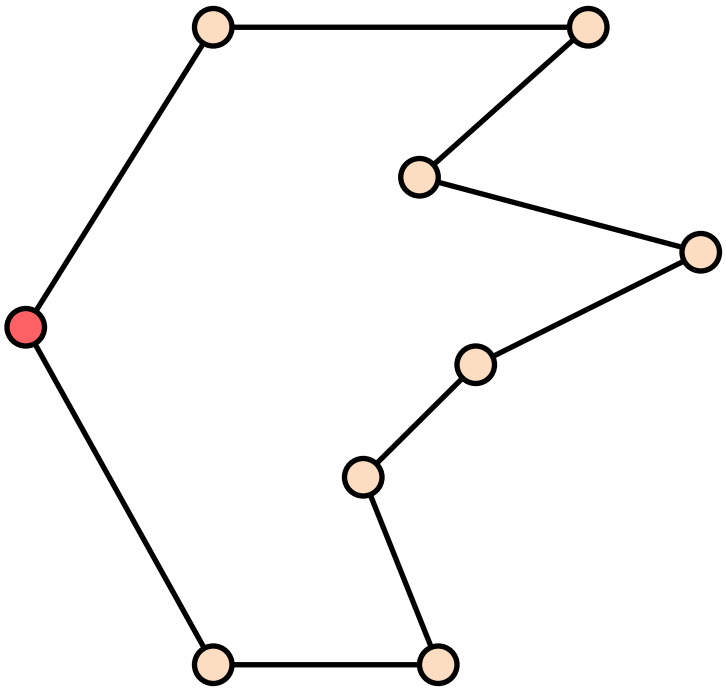
Graham's Scan



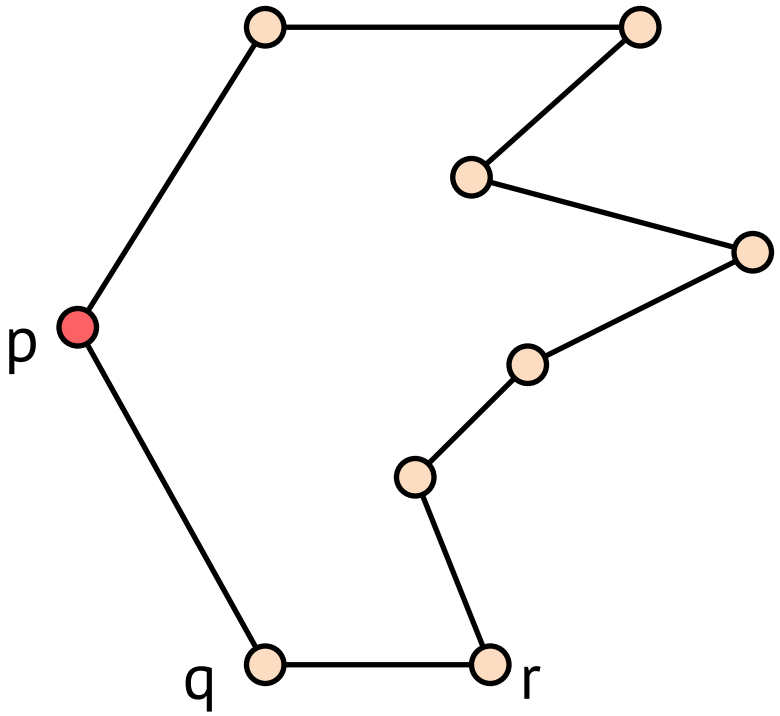
Graham's Scan



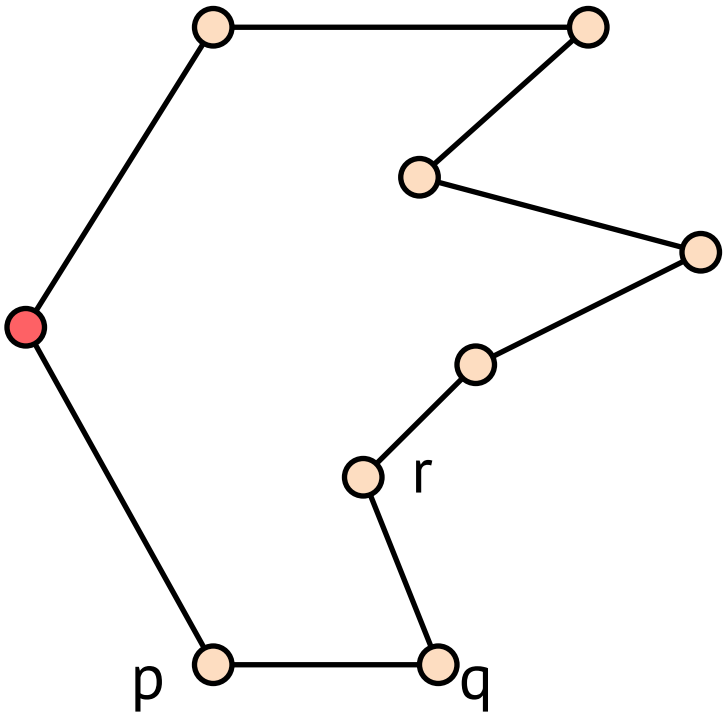
Graham's Scan



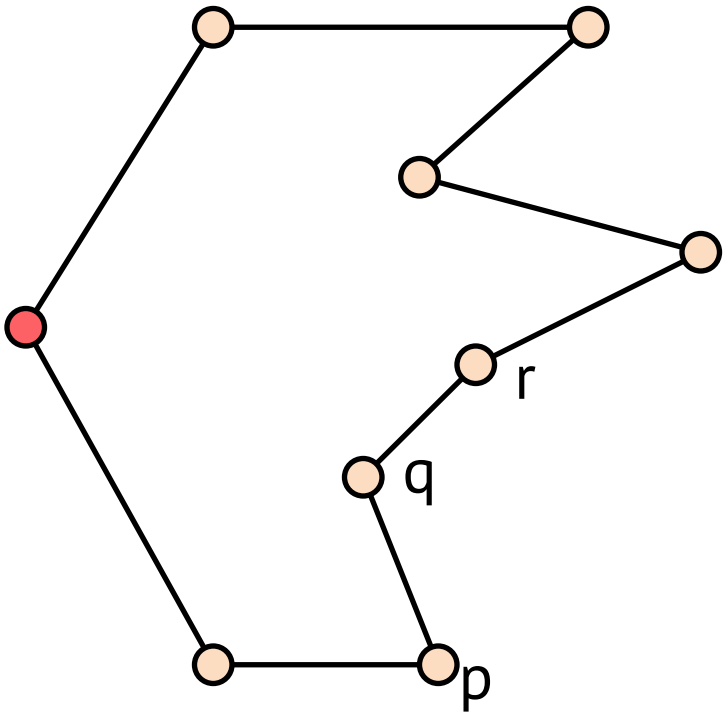
Graham's Scan



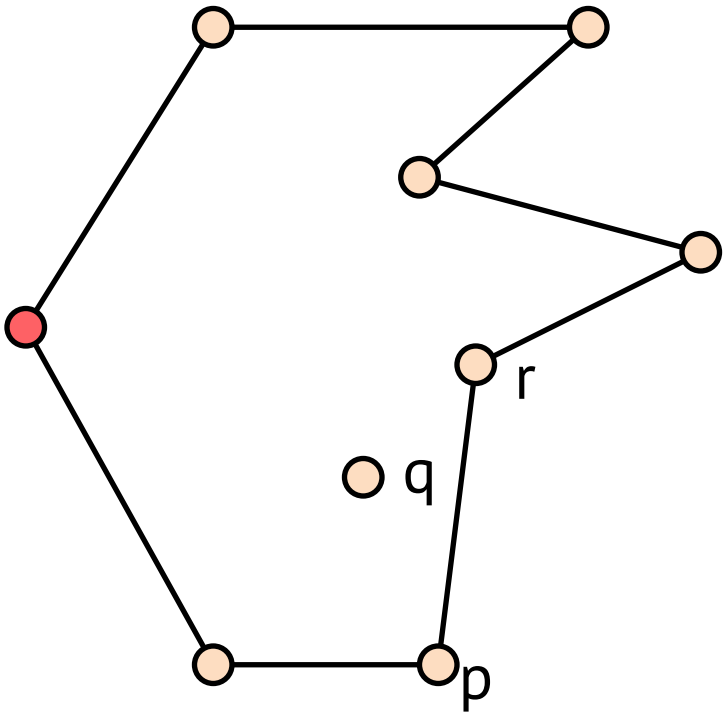
Graham's Scan



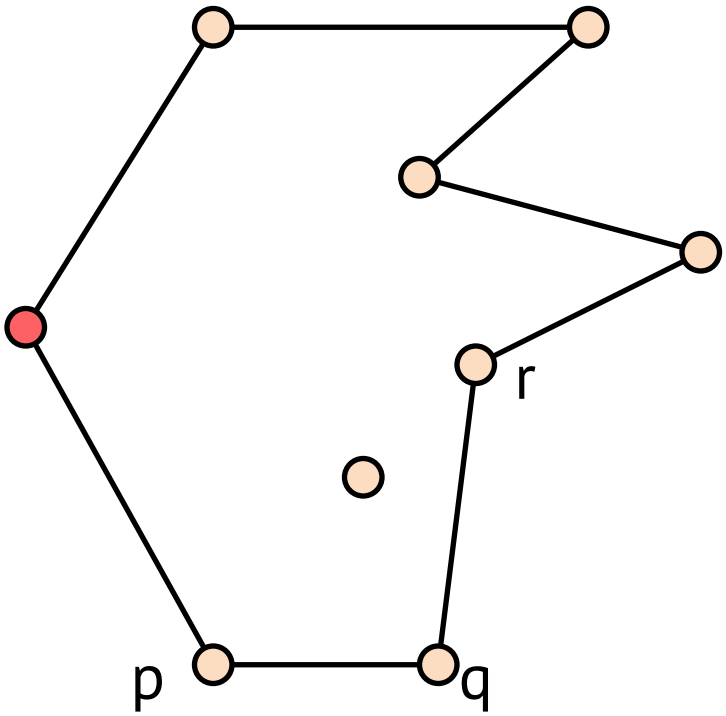
Graham's Scan



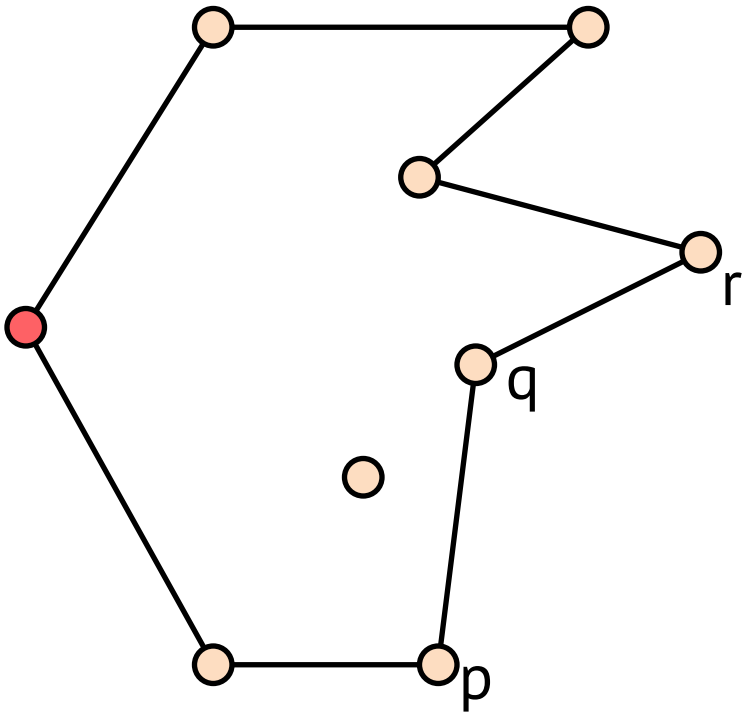
Graham's Scan



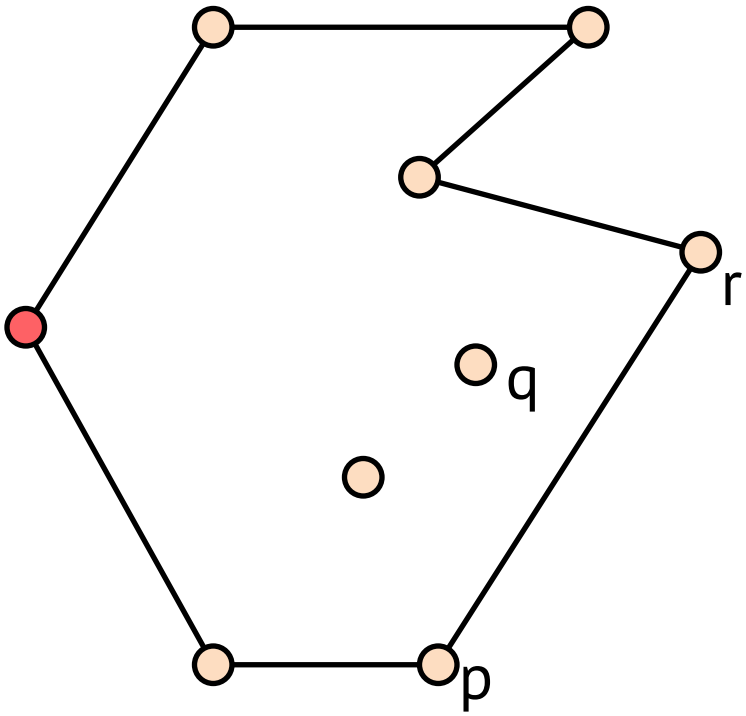
Graham's Scan



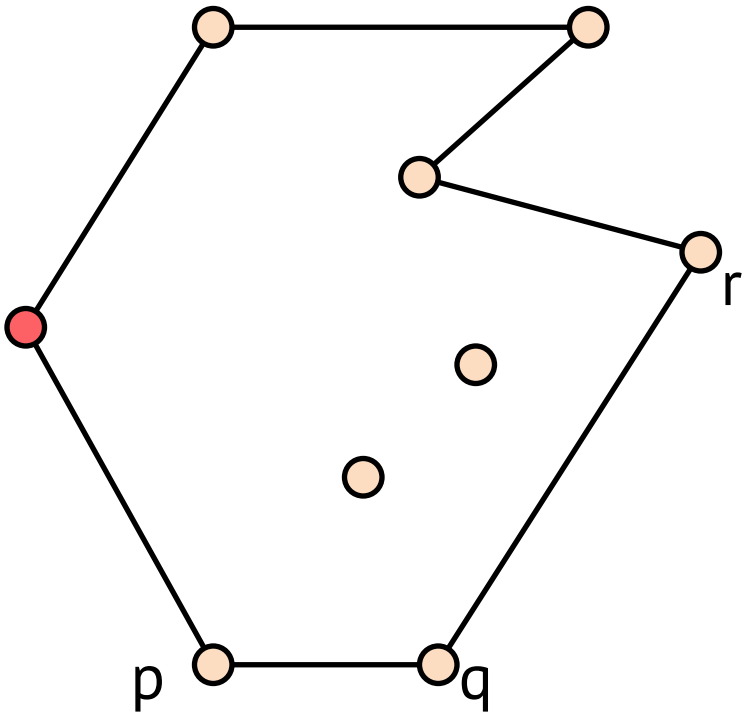
Graham's Scan



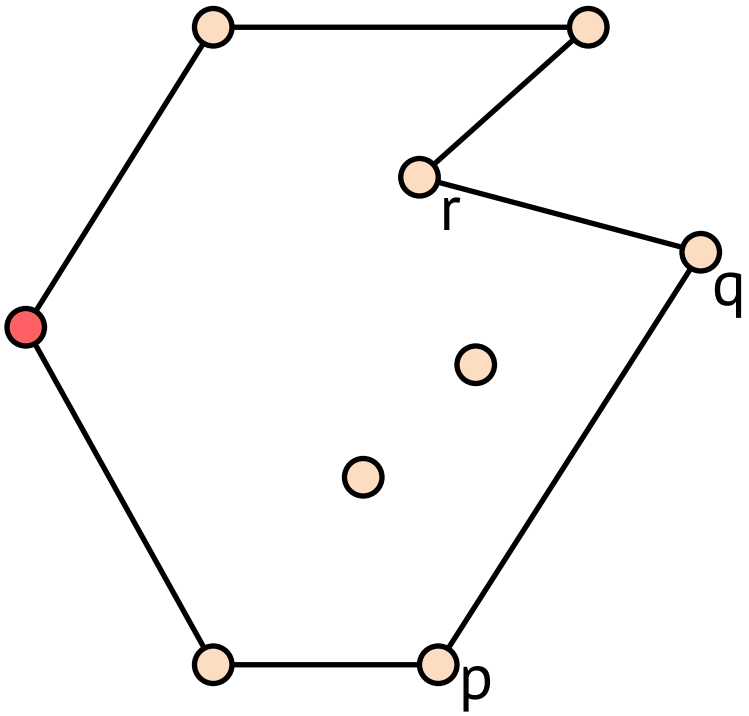
Graham's Scan



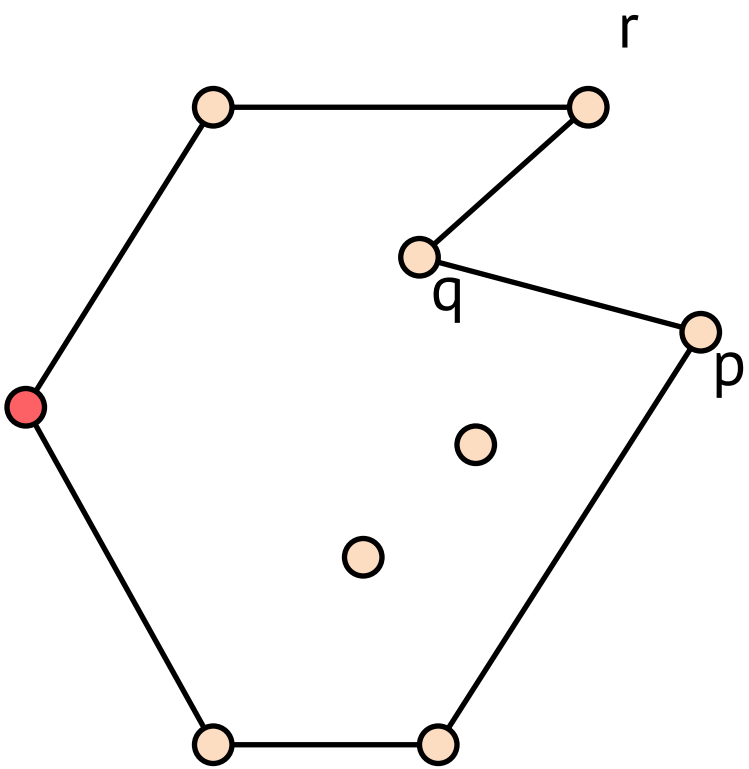
Graham's Scan



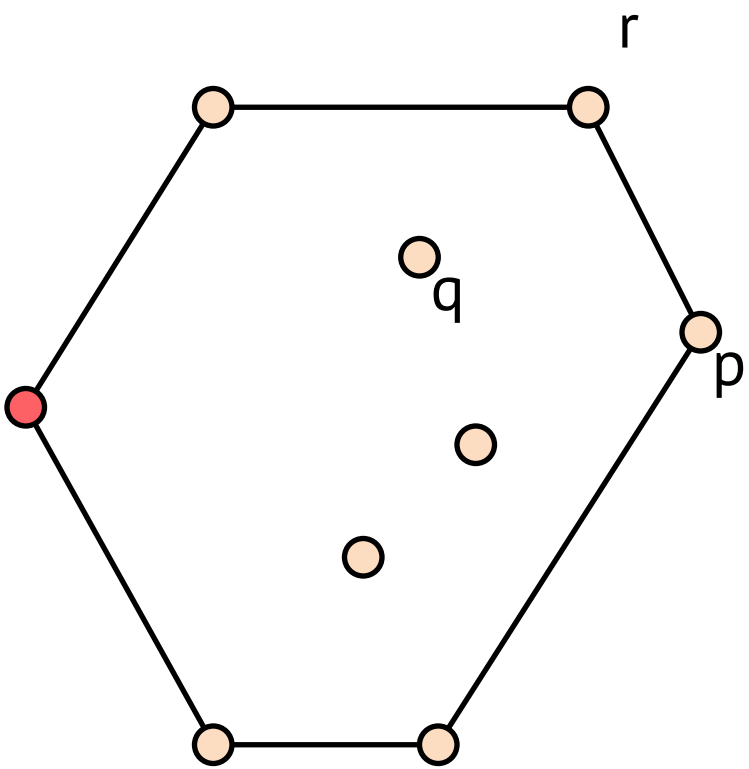
Graham's Scan



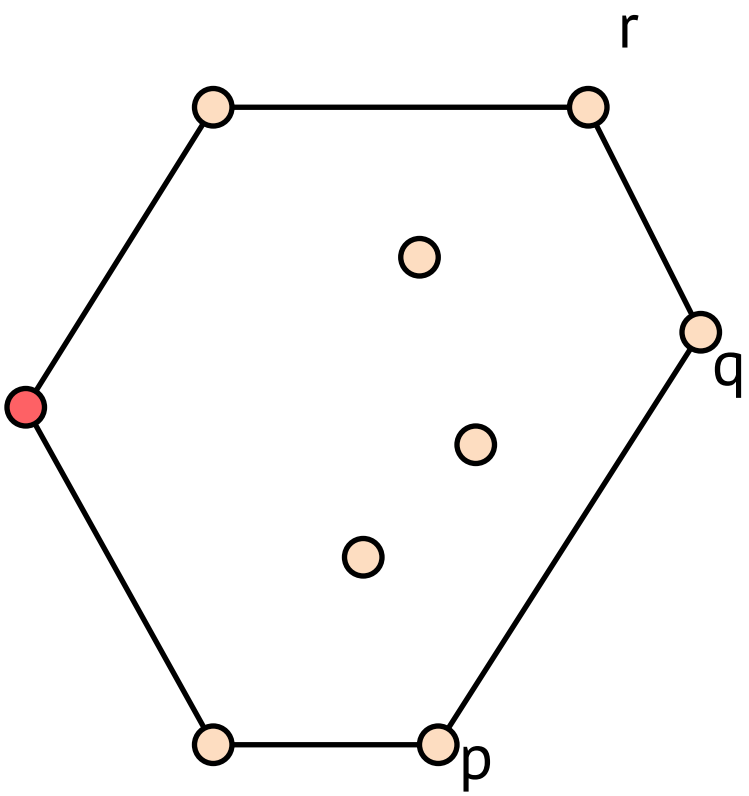
Graham's Scan



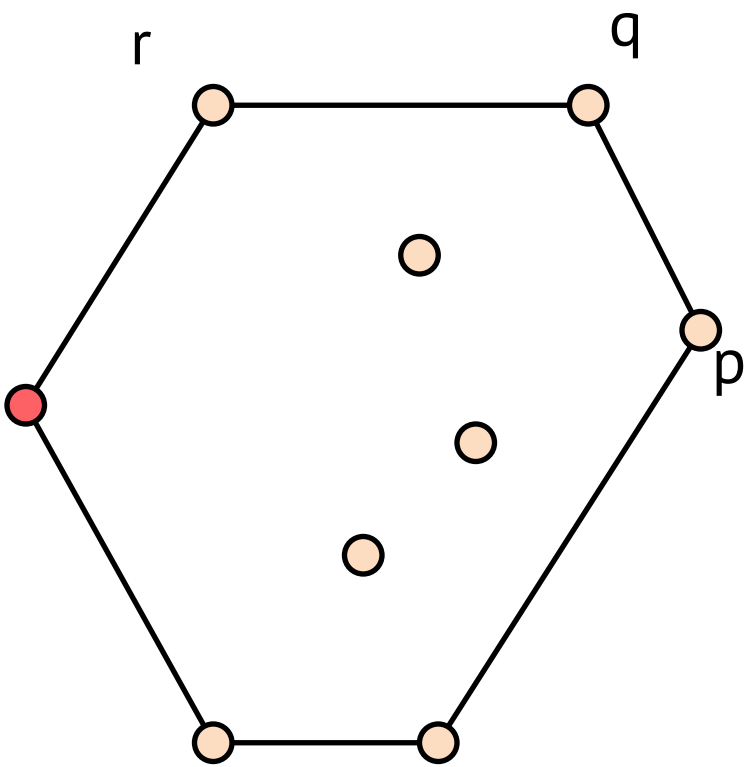
Graham's Scan



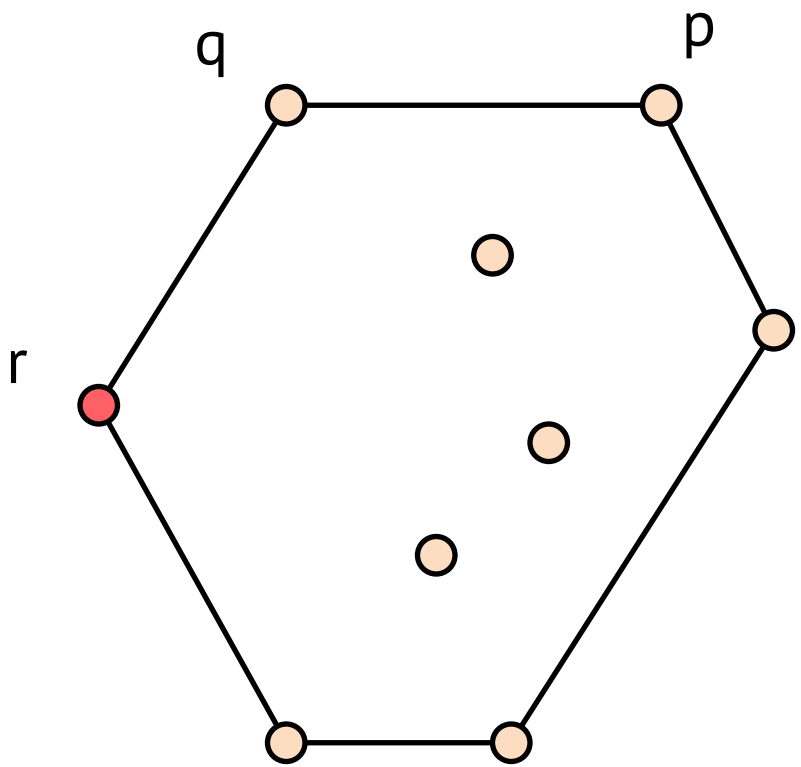
Graham's Scan



Graham's Scan



Graham's Scan

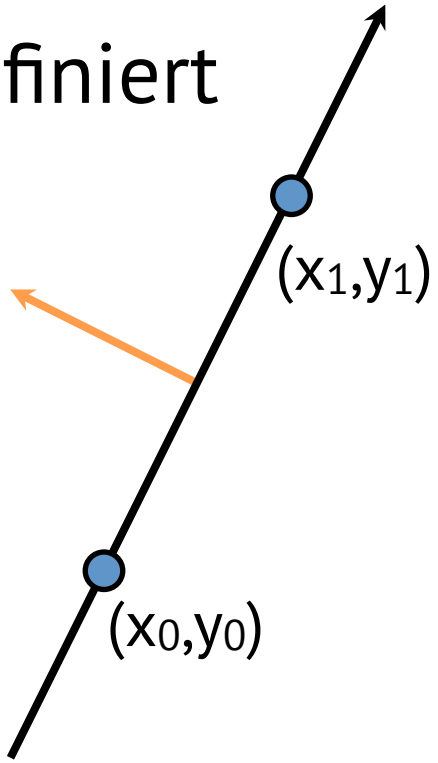


Halbraum-Test

- Es sei eine Supporting-Line durch die Punkte (x_0,y_0) und (x_1,y_1) gegeben
- Die Reihenfolge der Punkte definiert eine Orientierung
- Links ist innen, d.h. der Vektor

  (y_0-y_1,x_1-x_0)

  zeigt senkrecht nach innen



Skalarprodukt

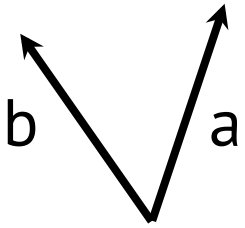
- Erinnerung: Skalarprodukt

  $(x_0,y_0)\cdot(x_1,y_1) = x_0 \times x_1 + y_0 \times y_1$

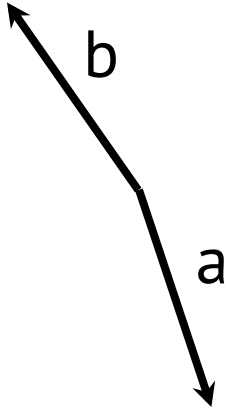
- Anschaulich:

$a \cdot b = 0$

$(\Leftrightarrow a \perp b)$



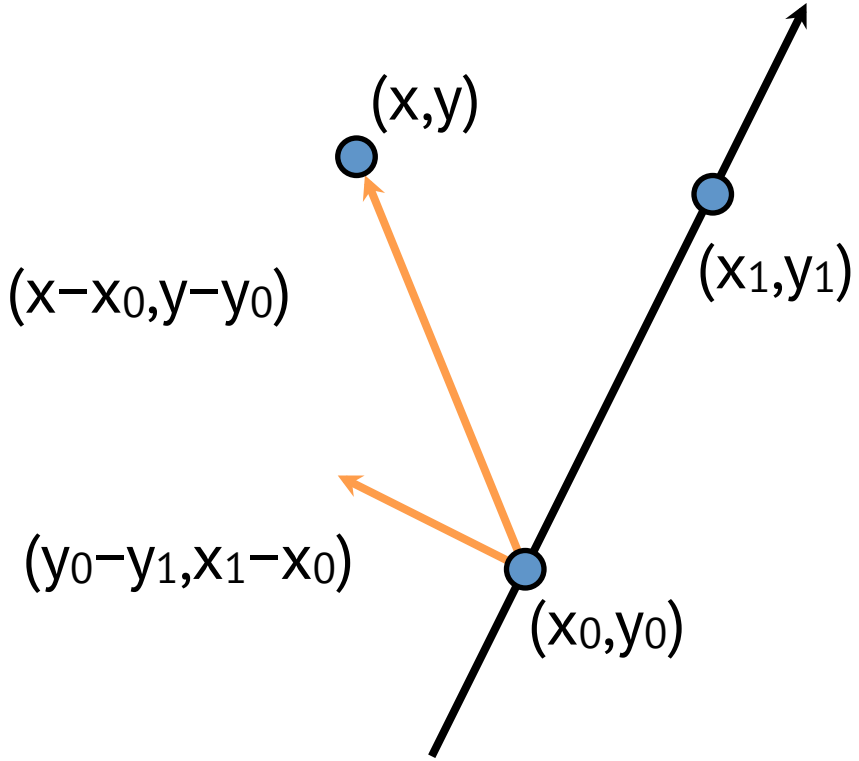
$a \cdot b > 0$



$a \cdot b < 0$

Halbraum-Test

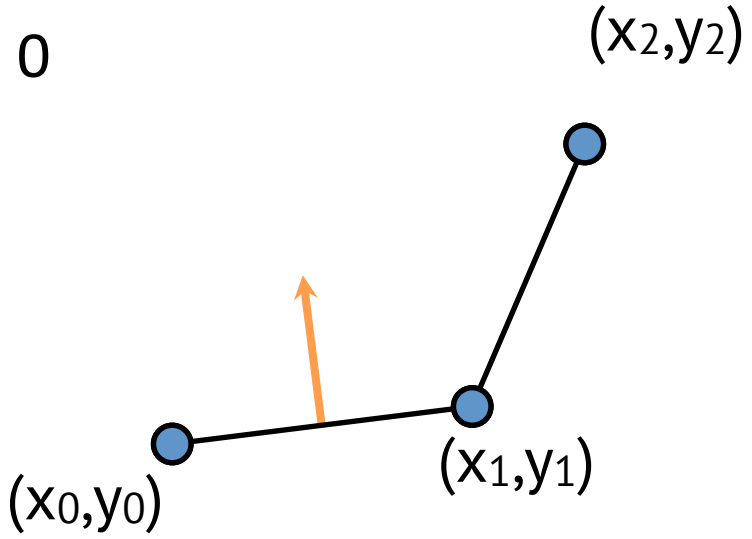
- Punkt (x,y) im Innern:
  $(x-x_0,y-y_0)\cdot(y_0-y_1,x_1-x_0) \geqslant 0$



Halbraum-Test

- Punkt (x_2,y_2) im Innern:
  $(x_2-x_0,y_2-y_0)\cdot(y_0-y_1,x_1-x_0) \geqslant 0$
  $(x_2-x_0)\times(y_0-y_1)+(y_2-y_0)\times(x_1-x_0) \geqslant 0$

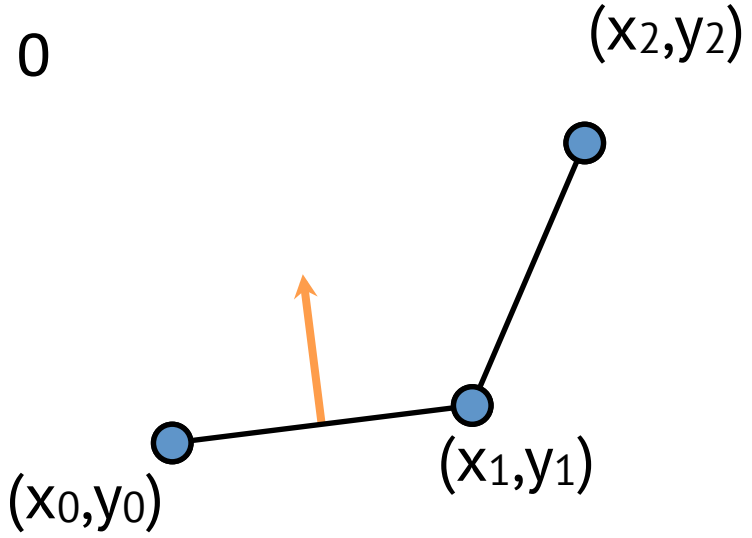
$$\det \begin{bmatrix} x_1-x_0 & x_2-x_0 \\ y_1-y_0 & y_2-y_0 \end{bmatrix} \geq 0$$



Halbraum-Test

- Punkt (x_2,y_2) im Innern:
  $(x_2-x_0,y_2-y_0)\cdot(y_0-y_1,x_1-x_0) \geqslant 0$
  $(x_2-x_0)\times(y_0-y_1)+(y_2-y_0)\times(x_1-x_0) \geqslant 0$

$$\det \begin{bmatrix} x_1-x_0 & x_2-x_1 \\ y_1-y_0 & y_2-y_1 \end{bmatrix} \geq 0$$



Analyse

- GrahamScan(pts)

```
let p be the left-most point in pts                          ] O(n)
sort pts by angle of line (p,pts[i]) to x-axis               ] O(n logn)
p ← head( pts ); q ← next(p); r ← next(q)
while r ≠ p do                                               ]
    if p,q,r is convex do                                    ]
        p ← next(p); q ← next(q); r ← next(r)                ] O(2n) =
    else                                                     ]   O(n)
        delete q                                             ]
        q ← p; p ← prev(p)                                   ]
```

Analyse

- Grahams Scan Algorithmus benötigt zur Berechnung der konvexen Hülle von n Punkten

$$O(n \times \log n)$$

2.7 Geometrische Algorithmen

Divide & Conquer

- Die konvexe Hülle von drei Punkten (in allg. Lage) ist ein Dreieck.
- Teile die Punkte in zwei Teilmengen mit disjunkter konvexer Hülle.
- Vereinige die konvexen Hüllen der Teilmengen.

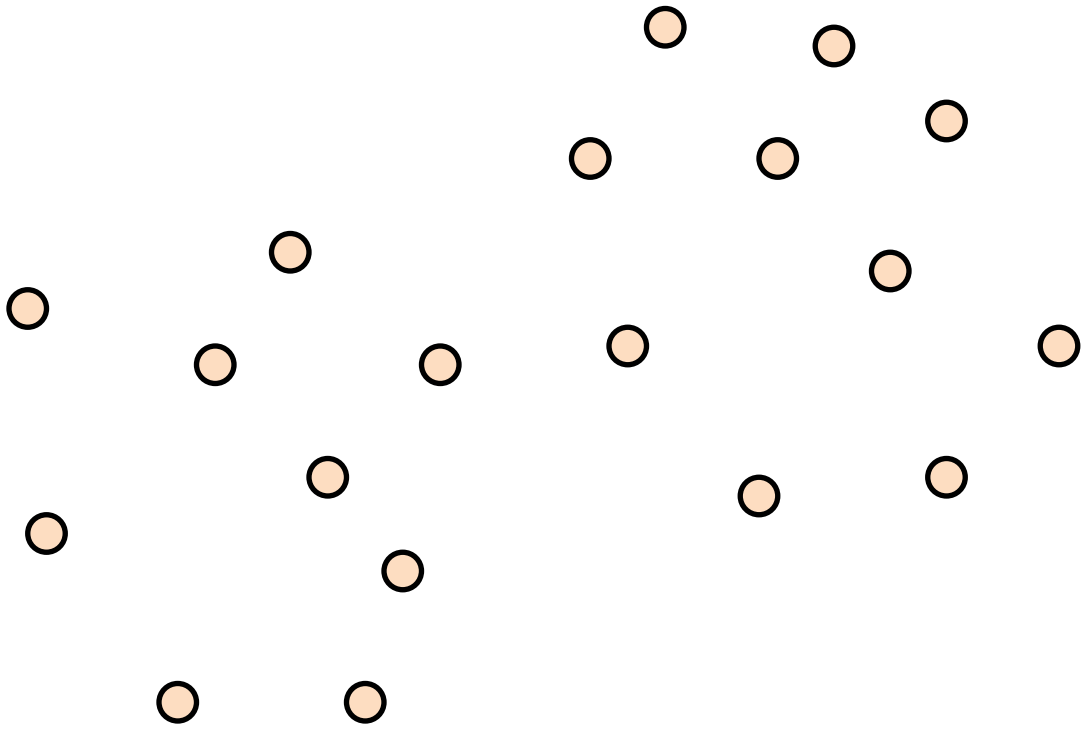
Divide & Conquer

- ConvexHull(pts[1..n])
  sort pts by increasing x-coordinate
  if x ⩾ 3 pts have same x-coordinate then
    remove interior points
  ConvexHullRec(pts[1..n])

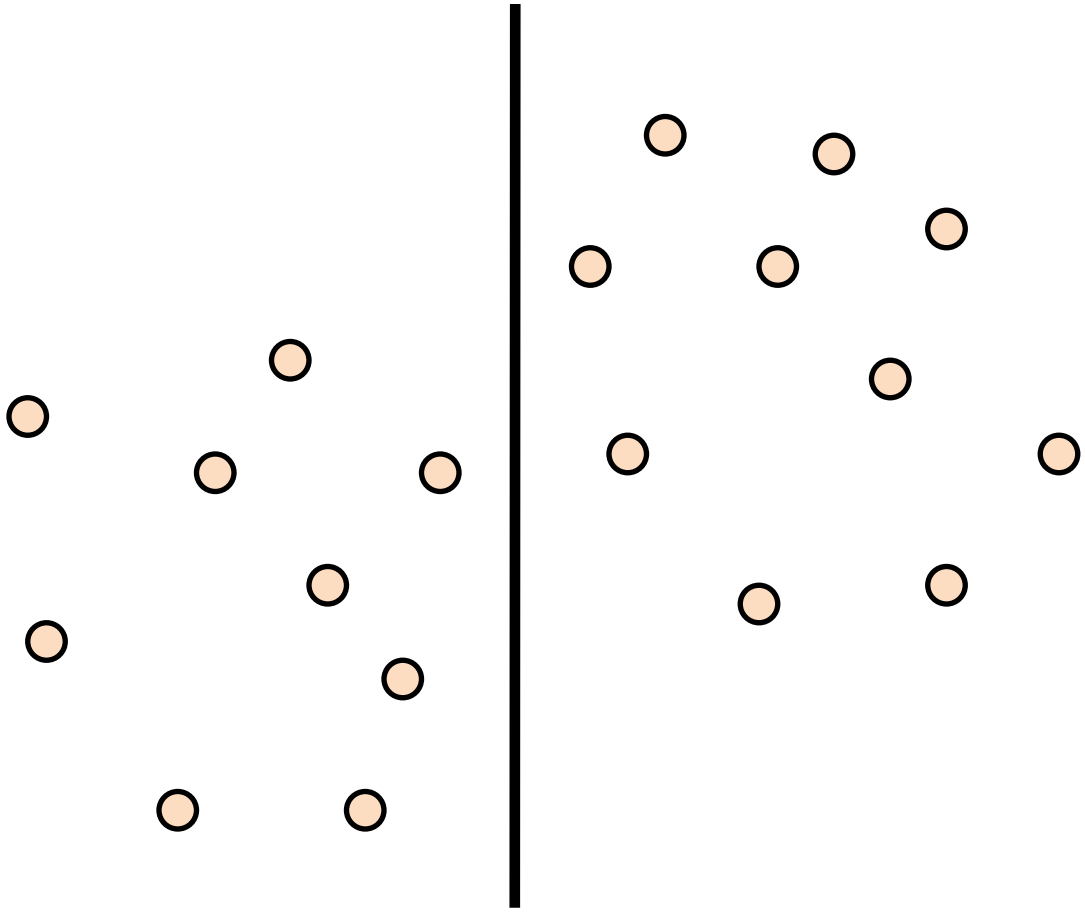
Divide & Conquer

- ConvexHullRec(pts[1..n])
  - p ← pts[1..n/2]
  - q ← pts[n/2+1..n]
  - $[p_1..p_m]$ ← ConvexHullRec(p)
  - $[q_1..q_n]$ ← ConvexHullRec(q)
  - find upper supporting line (p_i,q_j)
  - find lower supporting line (p_k,q_l)
  - return $[p_i,..,p_k,q_l,..q_j]$

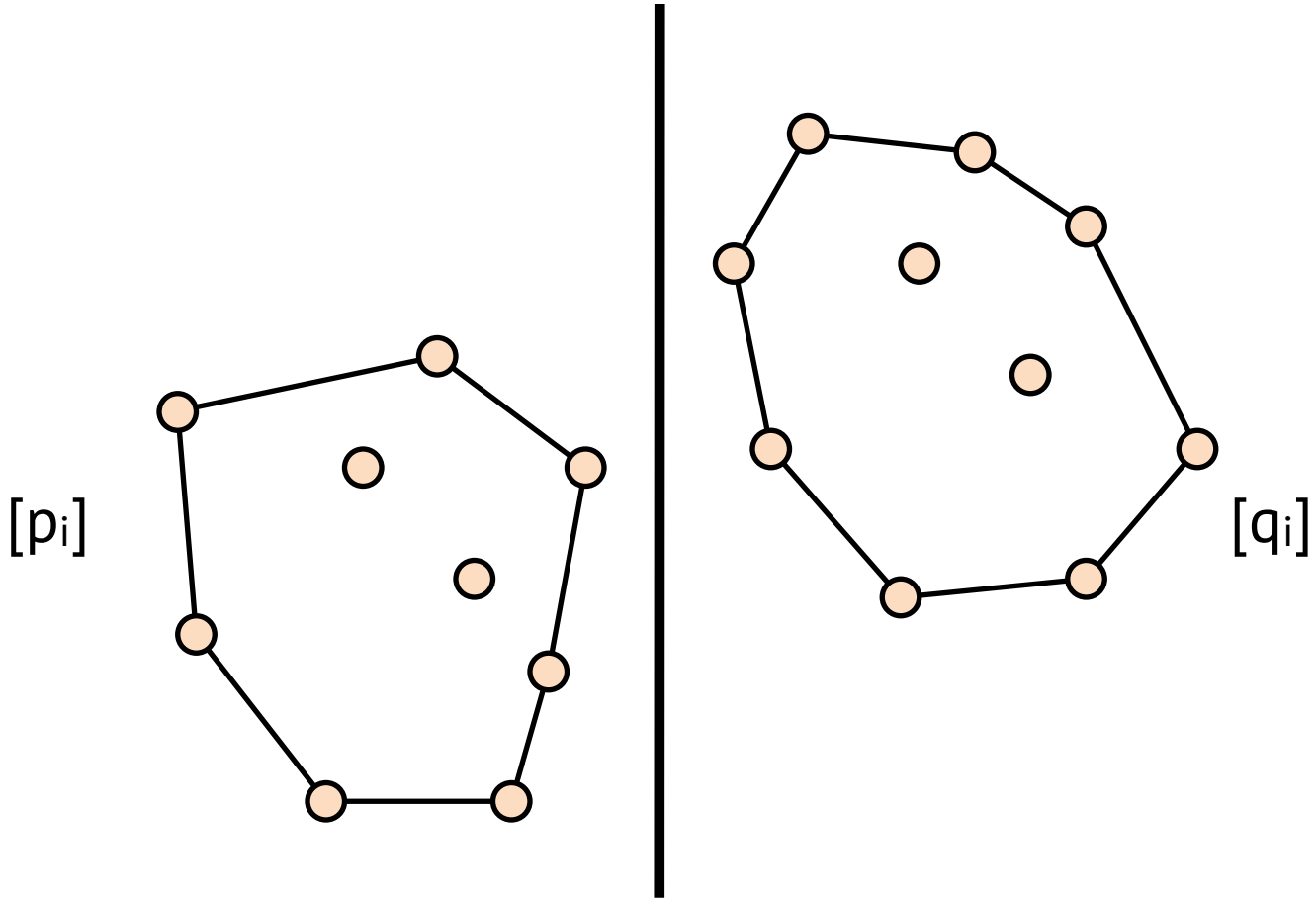
Divide & Conquer



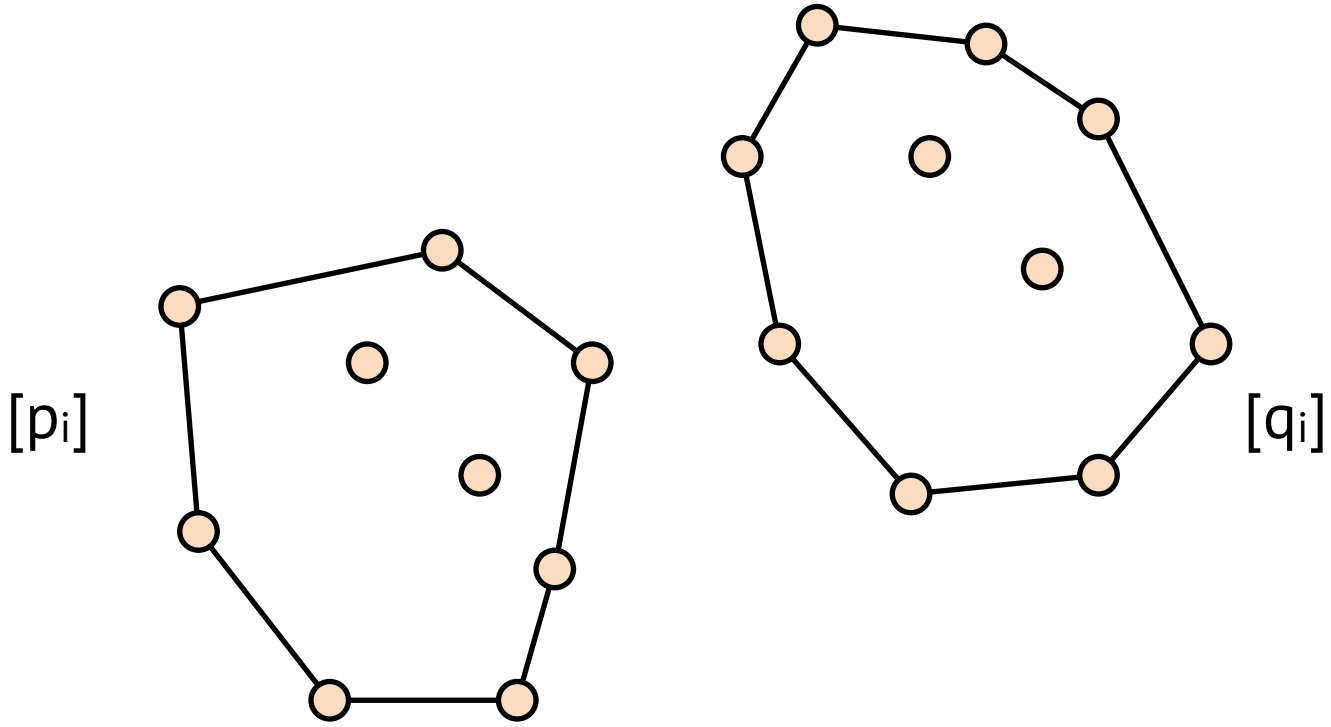
Divide & Conquer



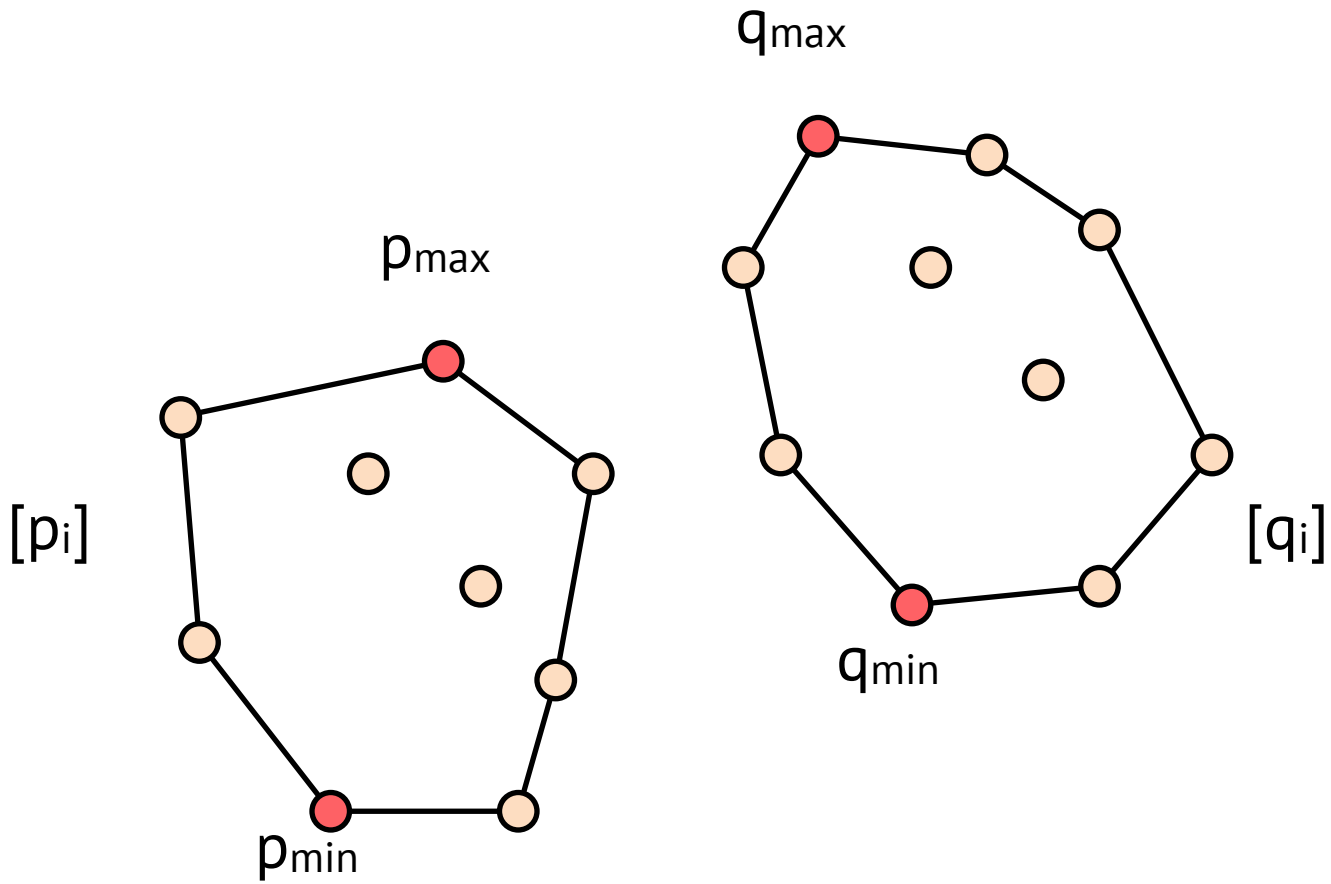
Divide & Conquer



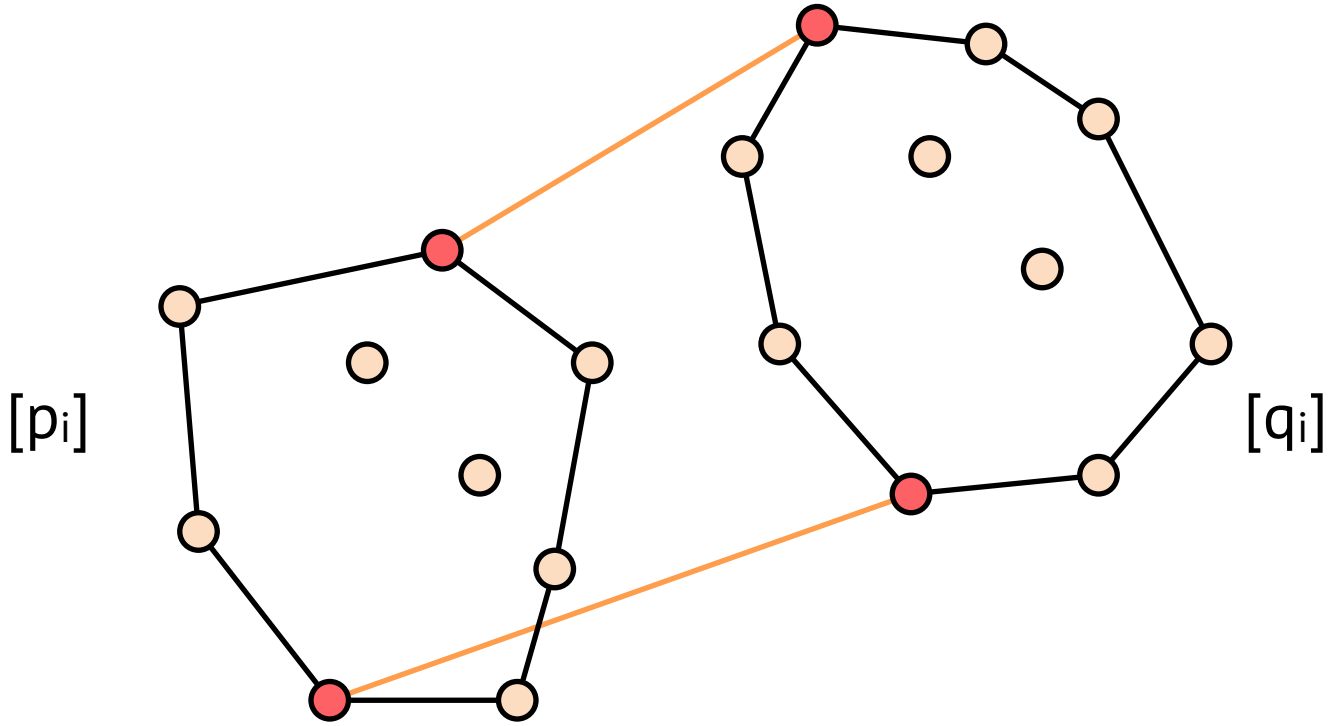
Divide & Conquer



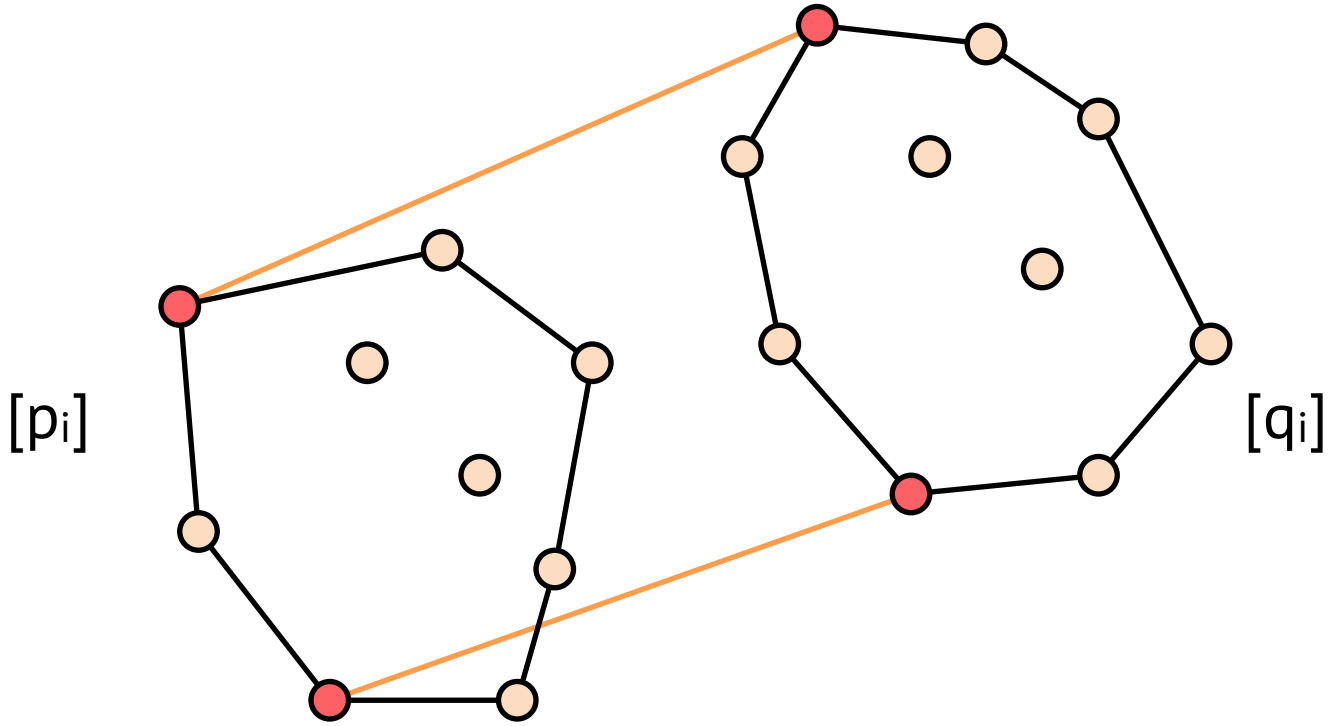
Divide & Conquer



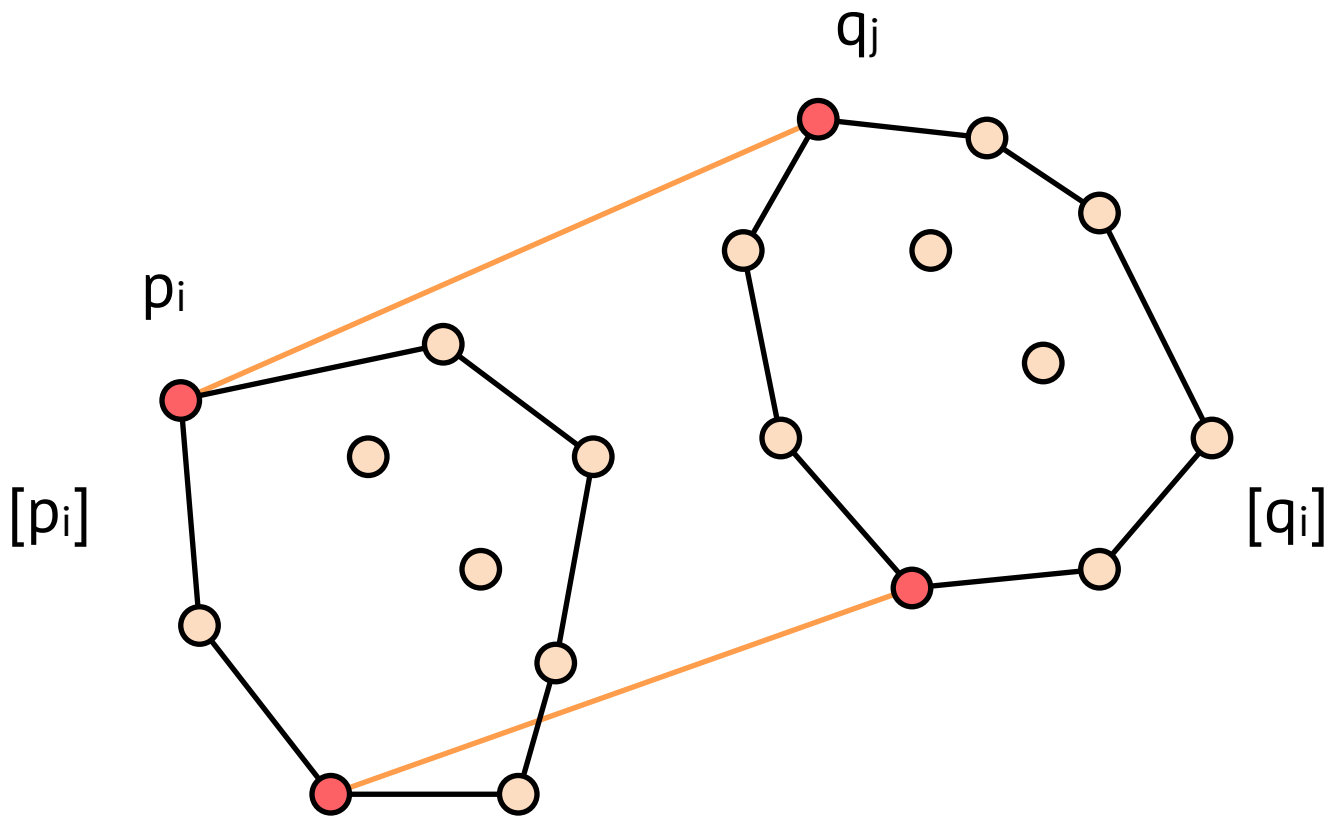
Divide & Conquer



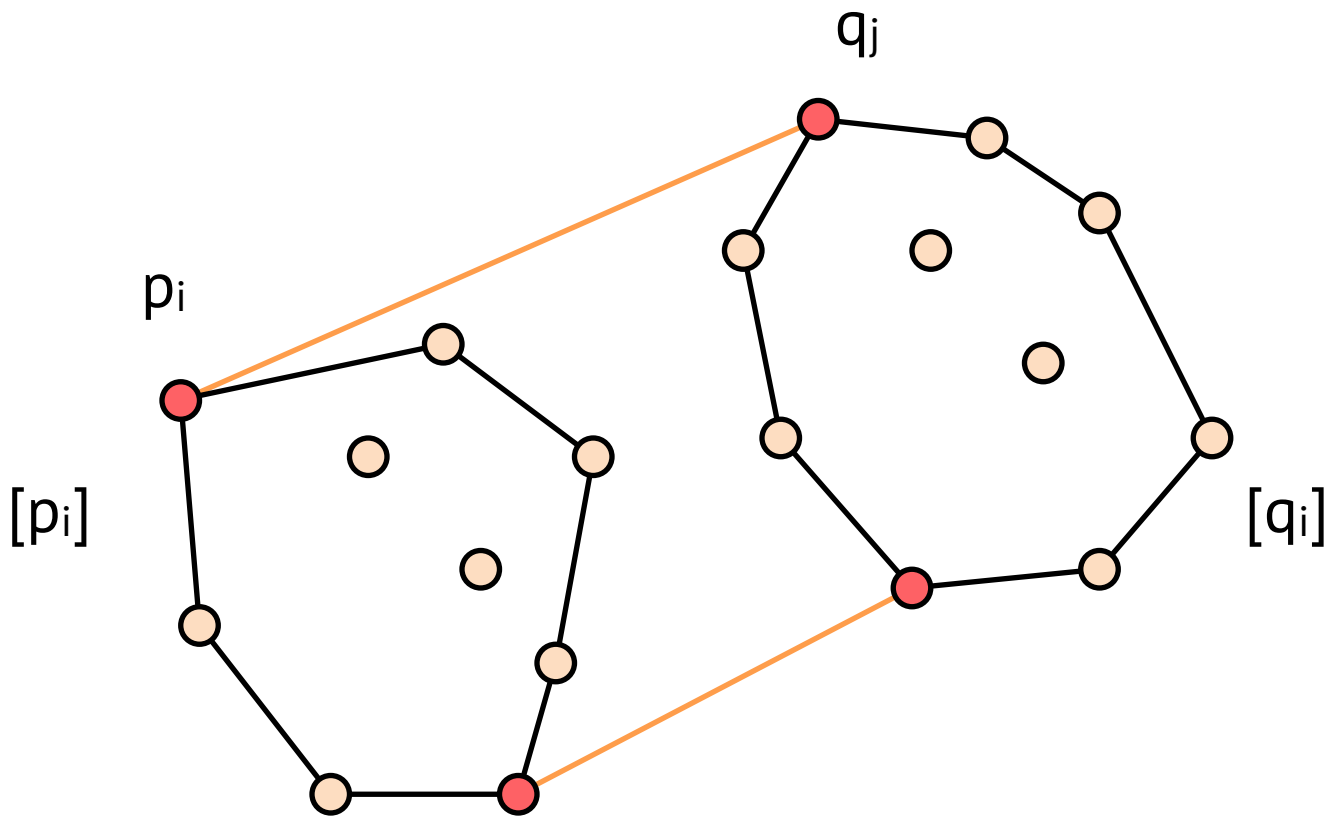
Divide & Conquer



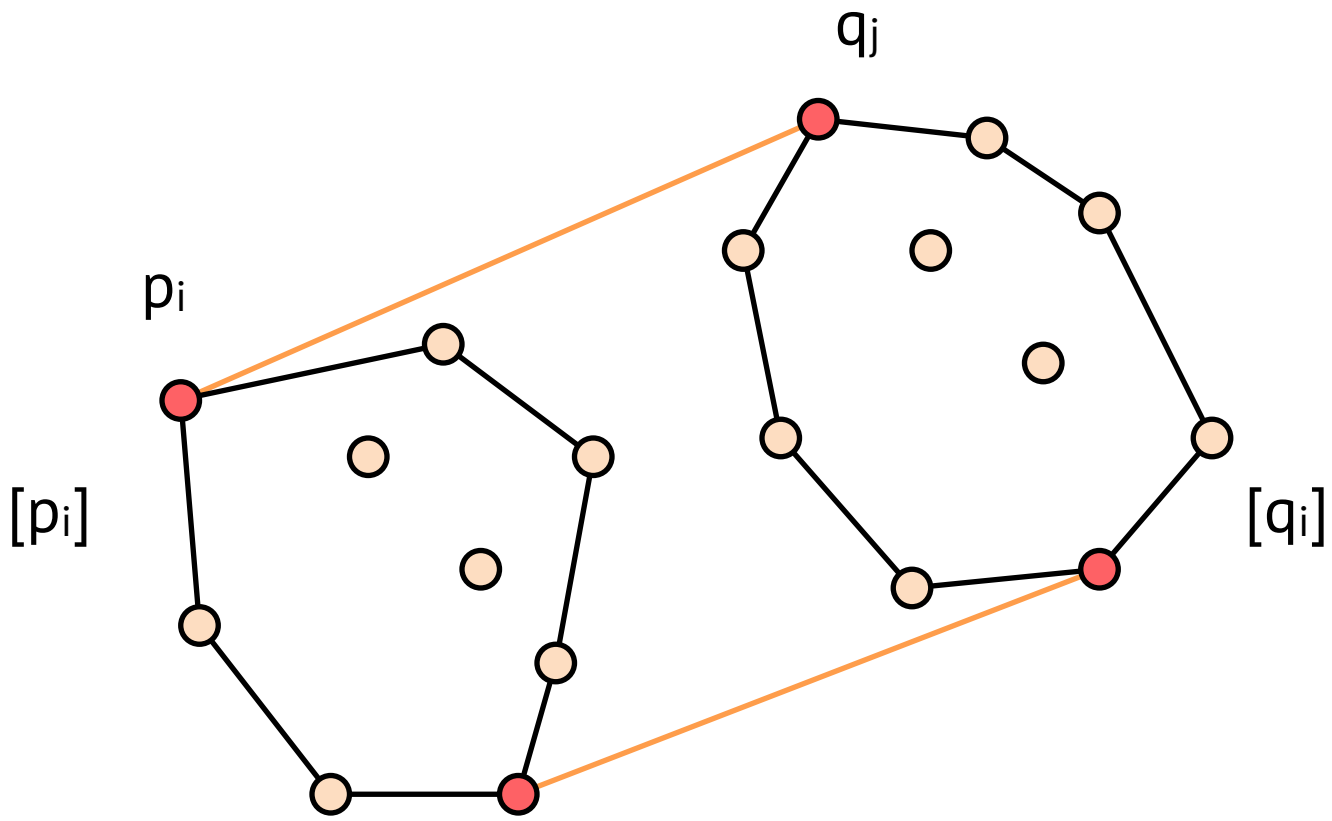
Divide & Conquer



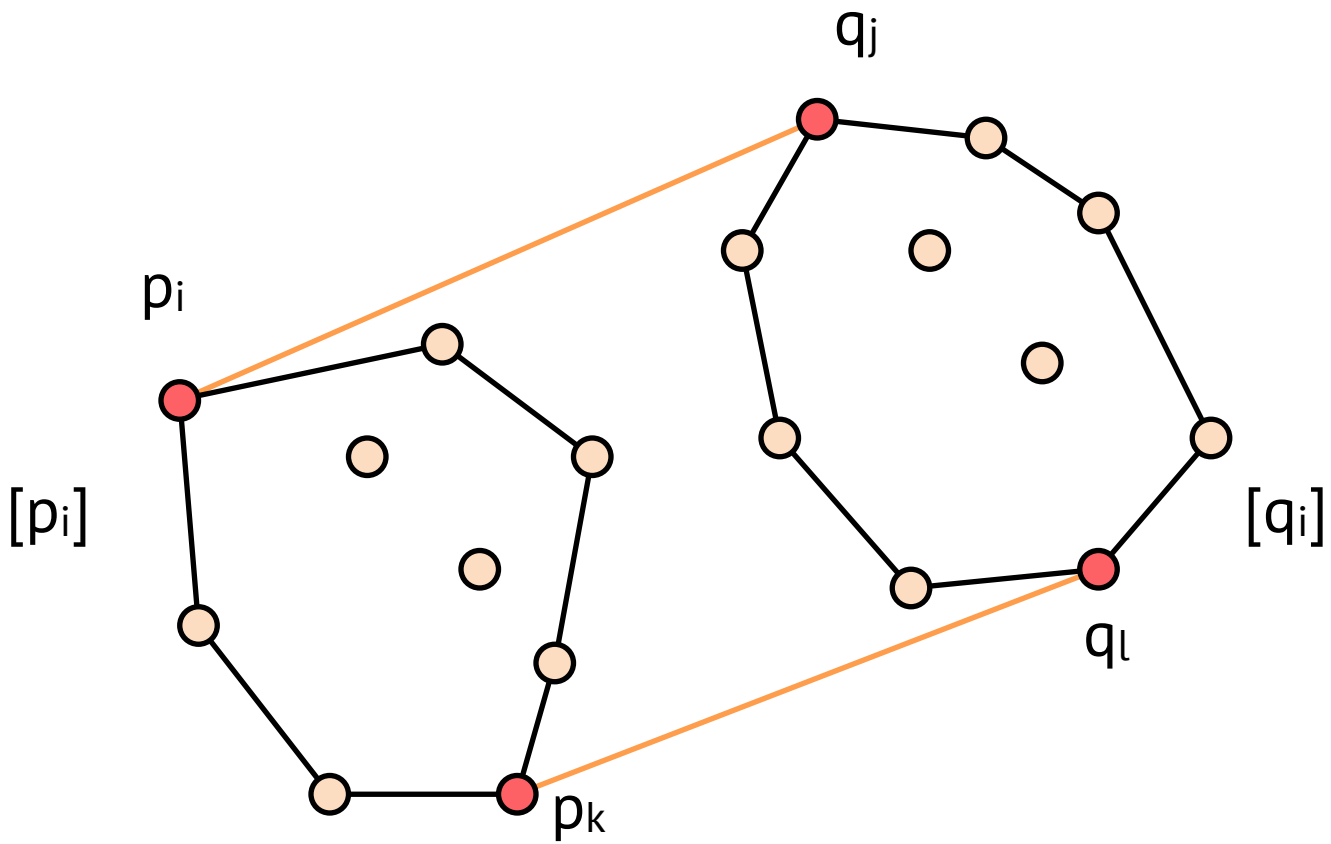
Divide & Conquer



Divide & Conquer

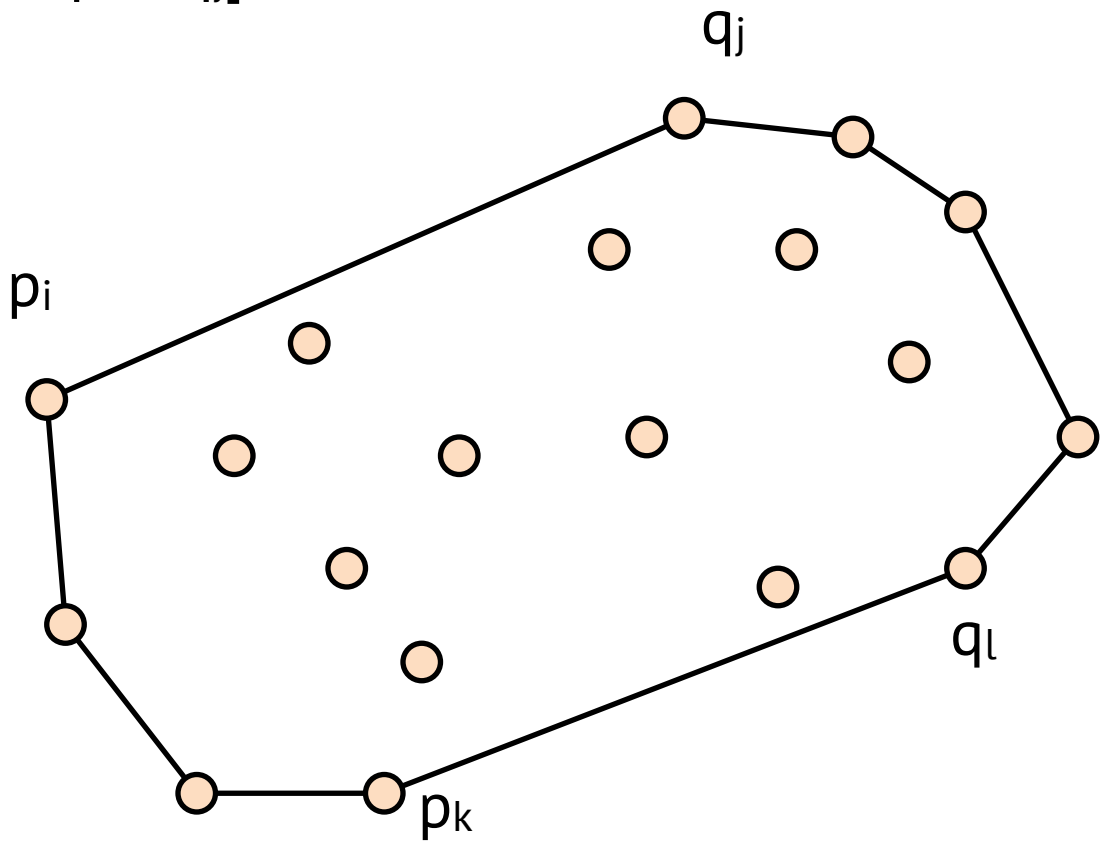


Divide & Conquer



Divide & Conquer

$[p_i,\ldots,p_k,q_l,\ldots,q_j]$



Supporting Lines

- Bestimme Punkte mit maximaler und minimaler y-Koordinate.

- Verschiebe die Supporting Line solange die entsprechende Ecke nicht konvex ist oder der Nachbarpunkt ausserhalb liegt.

Analyse

- ConvexHull(pts[1..n])
  - sort pts by increasing x-coordinate — O(nlogn)
  - if x ⩾ 3 pts have same x-coordinate then — O(n)
    - remove interior points
  - ConvexHullRec(pts[1..n])

Analyse

- ConvexHullRec(pts[1..n])
  - p ← pts[1..n/2]
  - q ← pts[n/2+1..n]
  - $[p_1..p_m]$ ← ConvexHullRec(p)
  - $[q_1..q_n]$ ← ConvexHullRec(q)
  - find upper supporting line (p_i,q_j)
  - find lower supporting line (p_k,q_l)
  - return $[p_i,..,p_k,q_l,..q_j]$

O(n)

Analyse

- $T(n) = 2\,T(n/2) + O(n)$
- Master-Theorem: $T(n) = O(n \times \log n)$
- Vorverarbeitung: $O(n \times \log n)$
- Gesamtaufwand: $O(n \times \log n)$

2.7 Geometrische Algorithmen

Optimaler Algorithmus

- Erinnerung: Ein optimaler Sortieralgorithmus hat beweisbar Aufwand O(n log n)
- Alle bisher vorgestellten Algorithmen zur Berechnung der konvexen Hülle benutzen einen Sortierschritt. Sie haben daher ebenfalls mindestens O(n logn).
- Gibt es einen besseren Algorithmus?

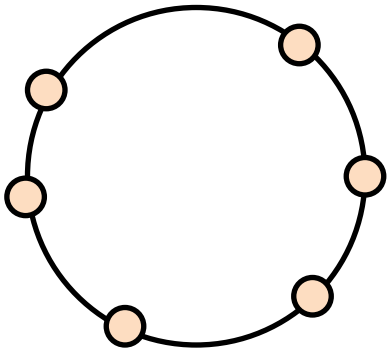
Optimaler Algorithmus

- Sei A ein optimaler CH-Algorithmus
- Verwende ihn zum Sortieren …
- Eingabe: $[v_1 \ldots v_n]$, oBdA $0 \leqslant v_i < 2\pi$
- Konvertierung: $v_i \longrightarrow p_i = (\cos v_i, \sin v_i)$
- CH-Polygon ergibt Reihenfolge

Optimaler Algorithmus

$[v_1 \ldots v_n]$

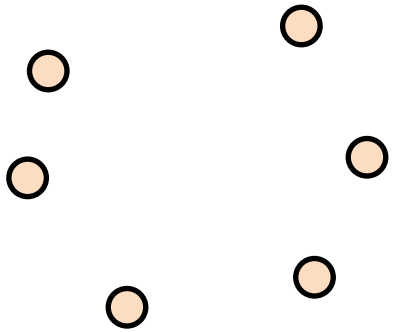
$[p_1 \ldots p_n]$



Optimaler Algorithmus

$[v_1 \ldots v_n]$

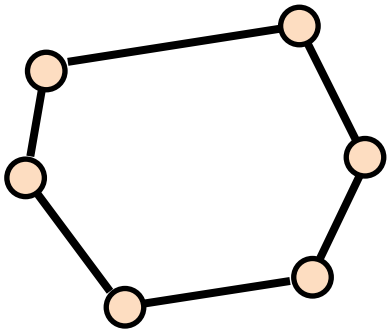
$[p_1 \ldots p_n]$



Optimaler Algorithmus

$[v_1 \ldots v_n]$

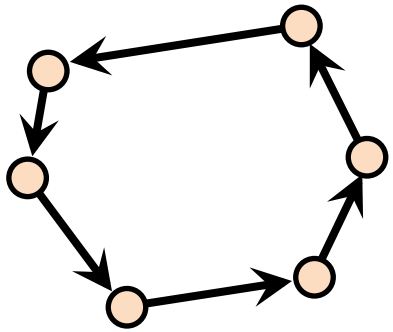
$[p_1 \ldots p_n]$



Optimaler Algorithmus

$[v_1 \ldots v_n]$

$[p_1 \ldots p_n]$



Optimaler Algorithmus

- Angenommen A ist ein CH-Algorithmus, der schneller als O(n logn) ist.
- Die Konvertierung $v_i \rightarrow p_i$ liefert einen Sortieralgorithmus, der schneller als O(n log n) ist.
- Dies ist ein Widerspruch zur Lower Bound von Sortieralgorithmen.