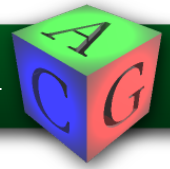
2.4 Suchen

2.4.1 Selektion

2.4.2 Hashing



2.4 Suchen

k-Selektion

- Selektions-Problem
  - gegeben:
    - n (verschiedene) Zahlen $A[1..n]$
    - eine Zahl k mit $1 \leqslant k \leqslant n$
  - gesucht: k-kleinstes Element von A, d.h. dasjenige Element $x \in A$ mit $\#\{ i : A[i] < x \} = k-1$

k-Selektion

- Man nennt das k-kleinste Element
  - für $k=1$ das Minimum
  - für $k=\lfloor (n+1)/2 \rfloor$, $k=\lceil (n+1)/2 \rceil$ den unteren, oberen Median
  - für $k=n$ das Maximum

k-Selektion: Beispiel

$$A[1..10] = 41, 76, 32, 62, 21, 52, 19, 83, 0, 91$$

Minimum: 0

Maximum: 91

Unterer Median: 41, denn

$$\#\{\ i : A[i] < 41\ \} = \{3, 5, 7, 9\} = 4 = \lfloor (10+1)/2 \rfloor - 1$$

Oberer Median: 52, denn

$$\#\{\ i : A[i] < 52\ \} = \{1, 3, 5, 7, 9\} = 5 = \lceil (10+1)/2 \rceil - 1$$

Selektion durch Sortieren

- SortSelect(A,k)

  sort(A)

  return A[k]

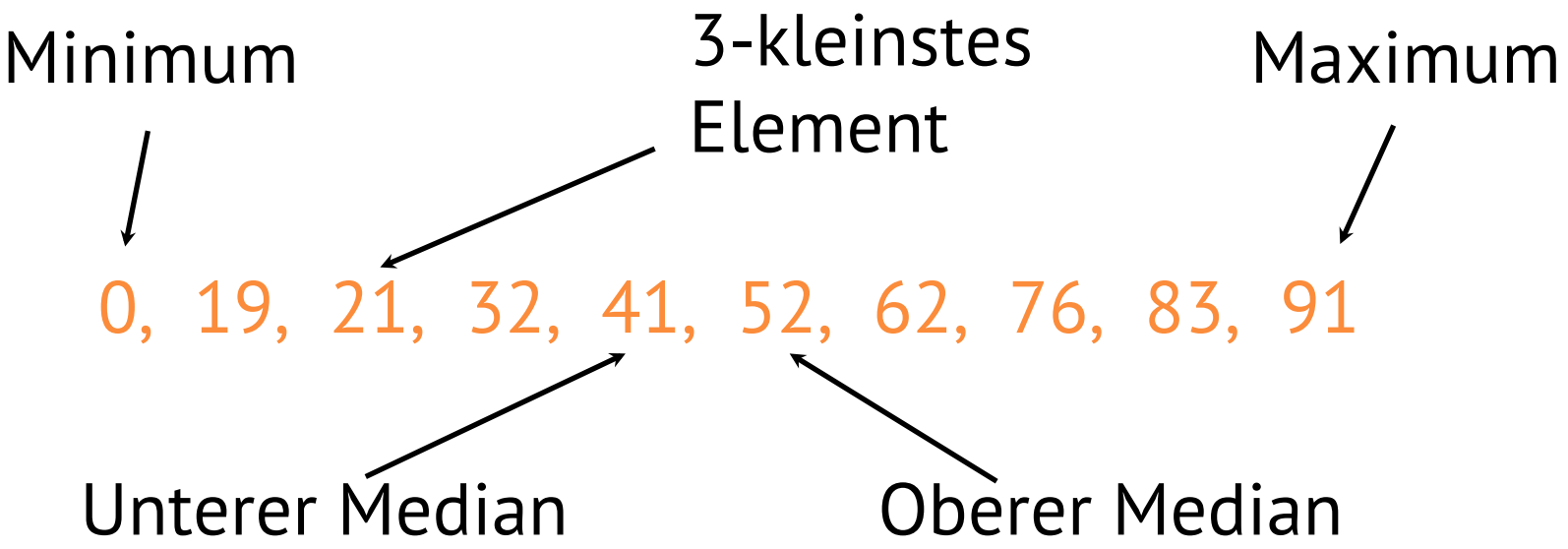
- Aufwand = Aufwand für Sortierung

  $T(n) = O(n \times \log n)$

k-Selektion: Beispiel

A[1..10] = 41, 76, 32, 62, 21, 52, 19, 83, 0, 91

Sortieren liefert:



2.4 Suchen

Selektion durch Partitionierung

- PartitionSelect(A[1..n], l, r, k)

```
if l = r then
    return A[ l ]
m ← Partition(A, l, r)
i ← m − l + 1
if k = i then
    return A[m]
elseif k < i then
    return PartitionSelect(A, l, m−1, k )
else
    return PartitionSelect(A,m+1,r,k−i )
```

Berechne das k-kleinste Element von A[l..r]

Precondition: $1 \leqslant k \leqslant r - l + 1$

Initialer Aufruf mit l=1, r=n

Selektion durch Partitionierung

- Aufwand
  - Worst case (vgl. Quick-Sort)

    $T(n) = O(n^2)$
  - Average case (ohne Beweis)

    $T(n) = O(n)$
  - anschaulich: $n/2 + n/4 + \ldots \leqslant n$

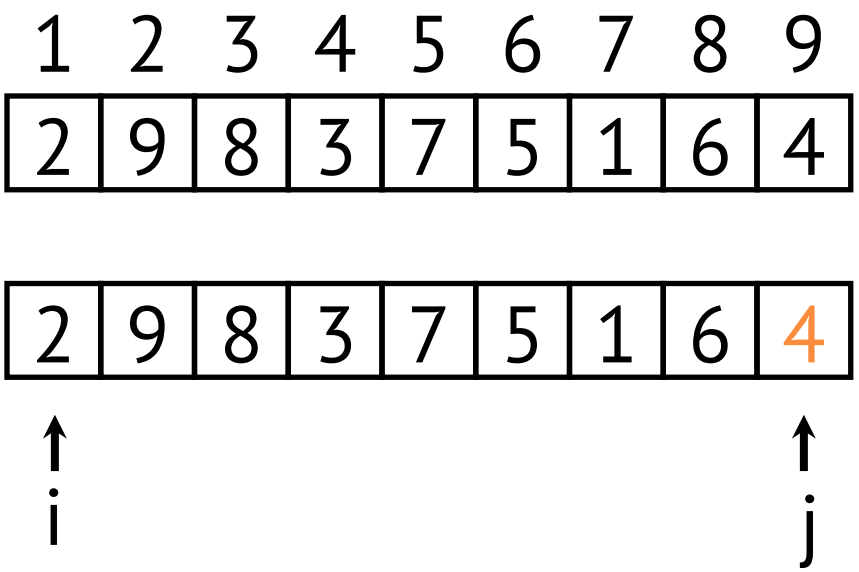
Beispiel

1	2	3	4	5	6	7	8	9
2	9	8	3	7	5	1	6	4

k = 6

Beispiel

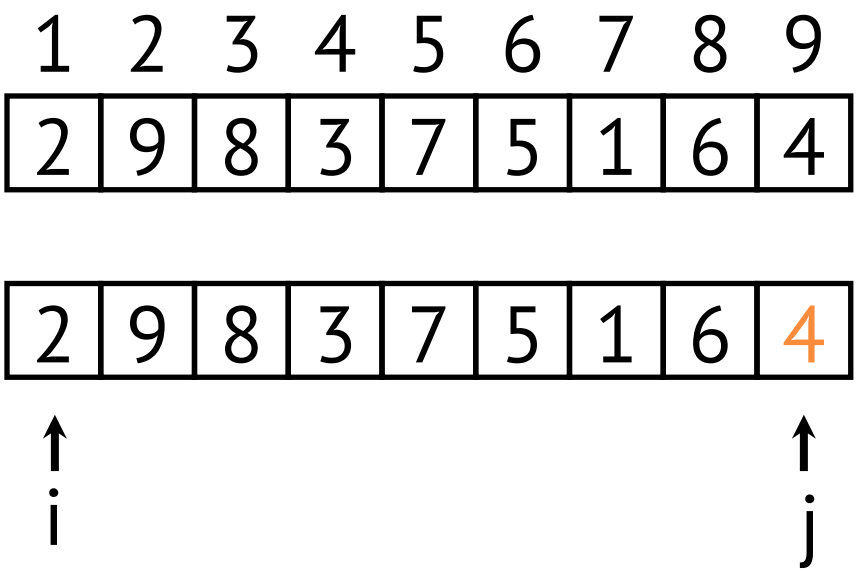
k = 6



Pivot = 4

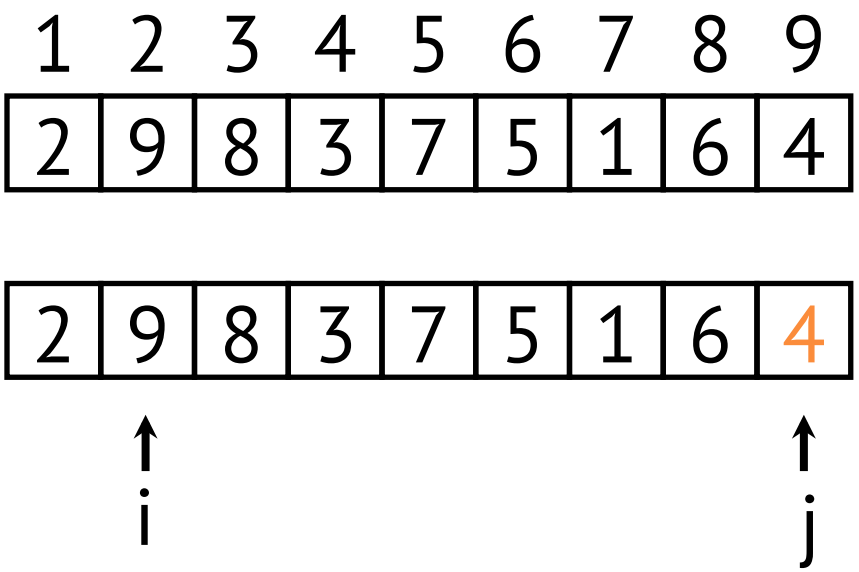
Beispiel

k = 6



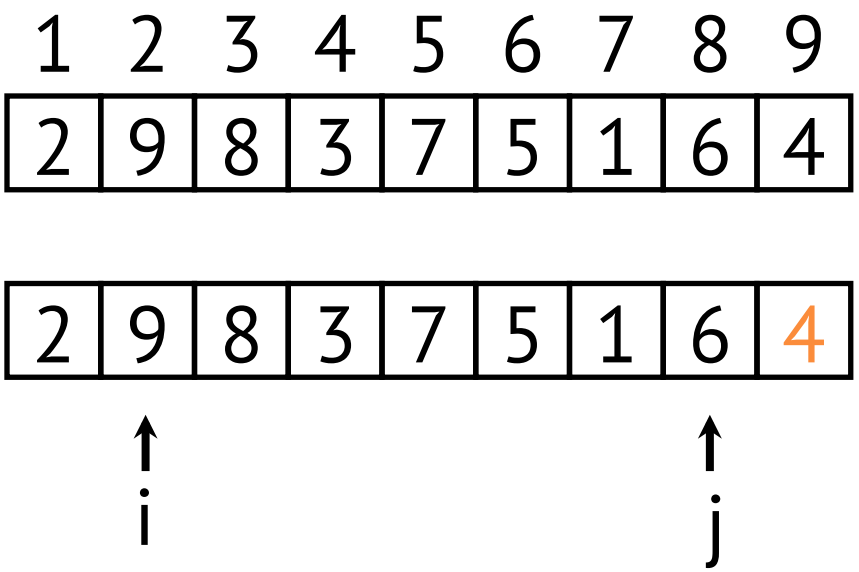
Beispiel

k = 6



Beispiel

k = 6



Beispiel

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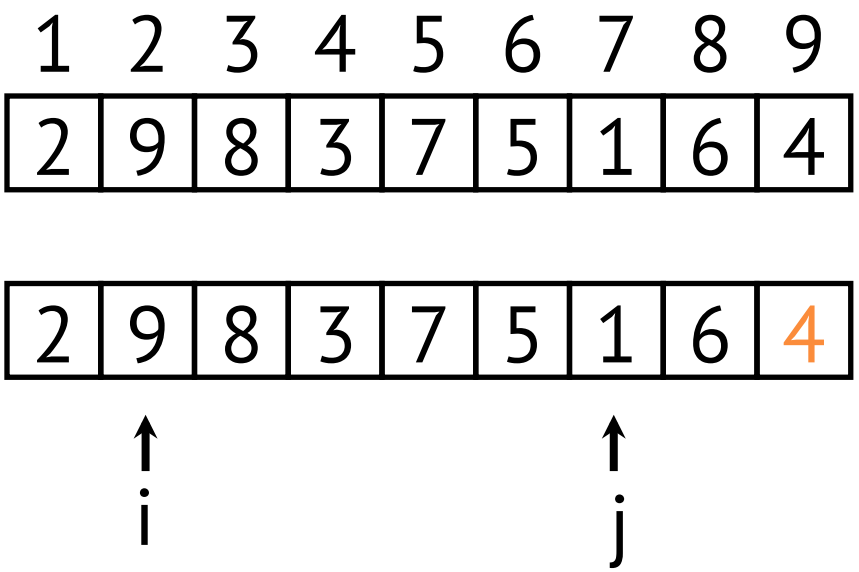
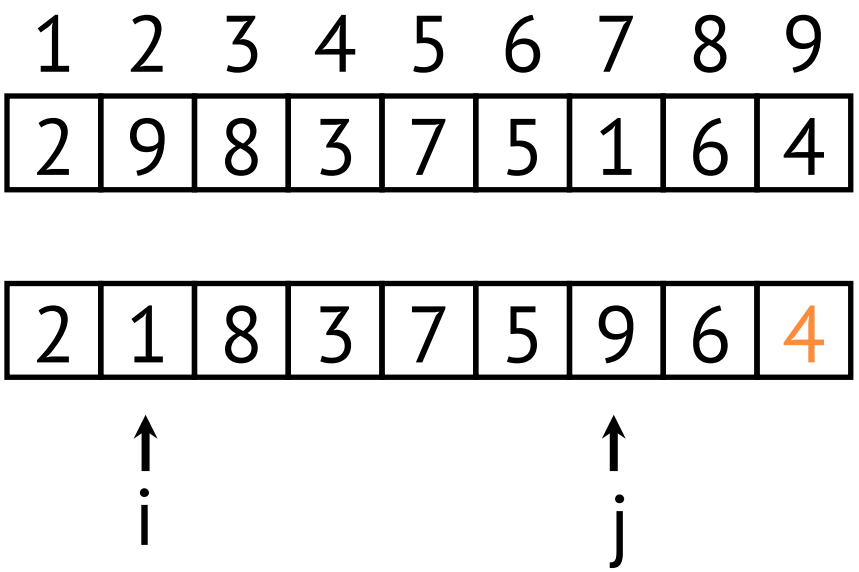


Beispiel

k = 6



Beispiel

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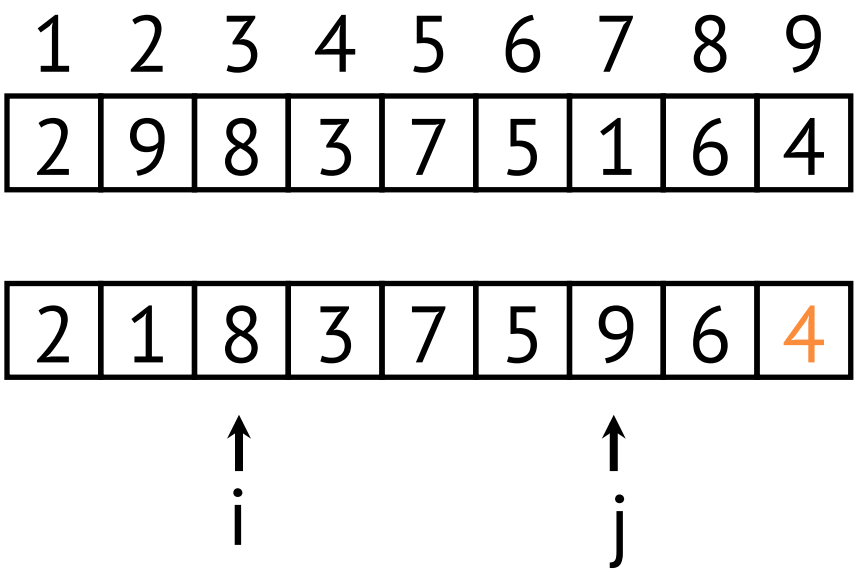
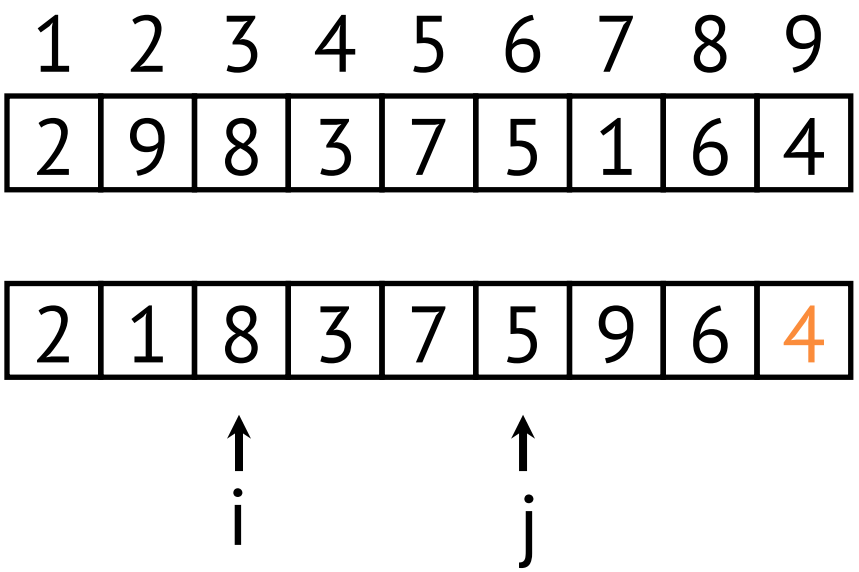


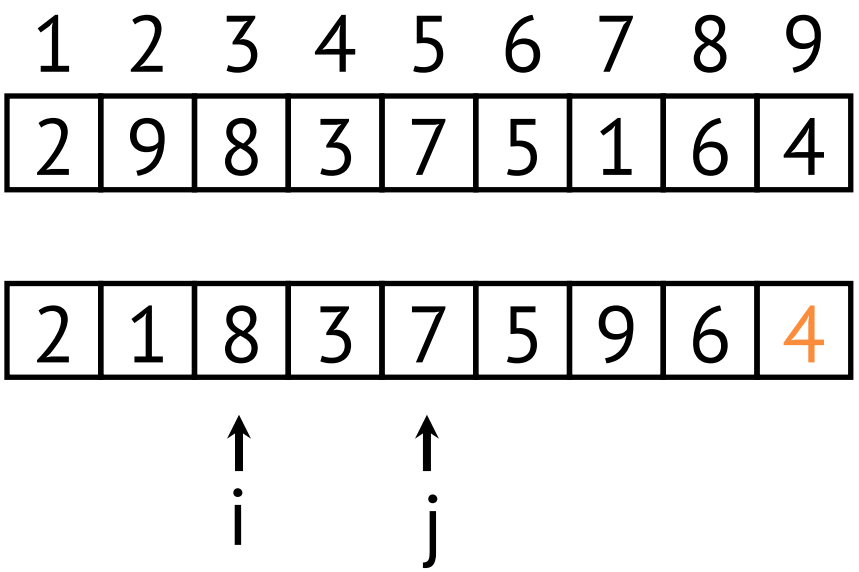
Beispiel

k = 6



Beispiel

k = 6



Beispiel

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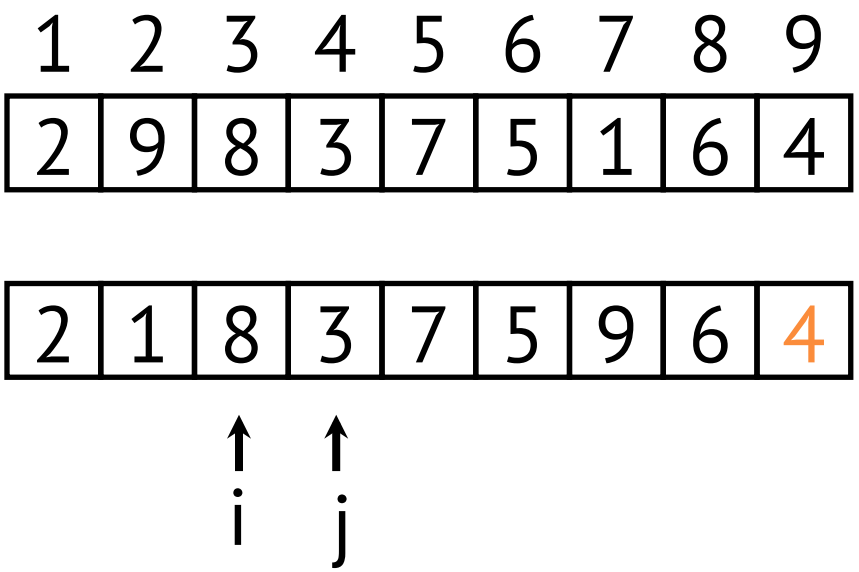

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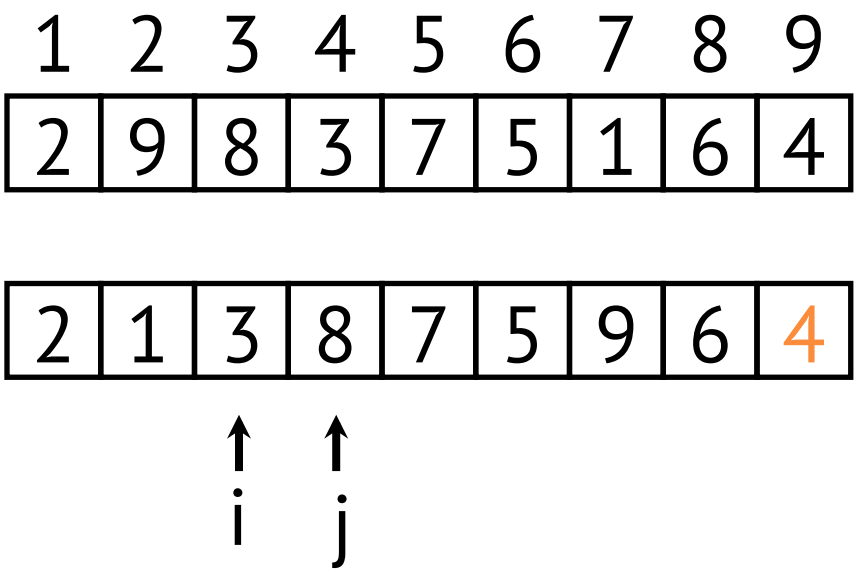

Beispiel

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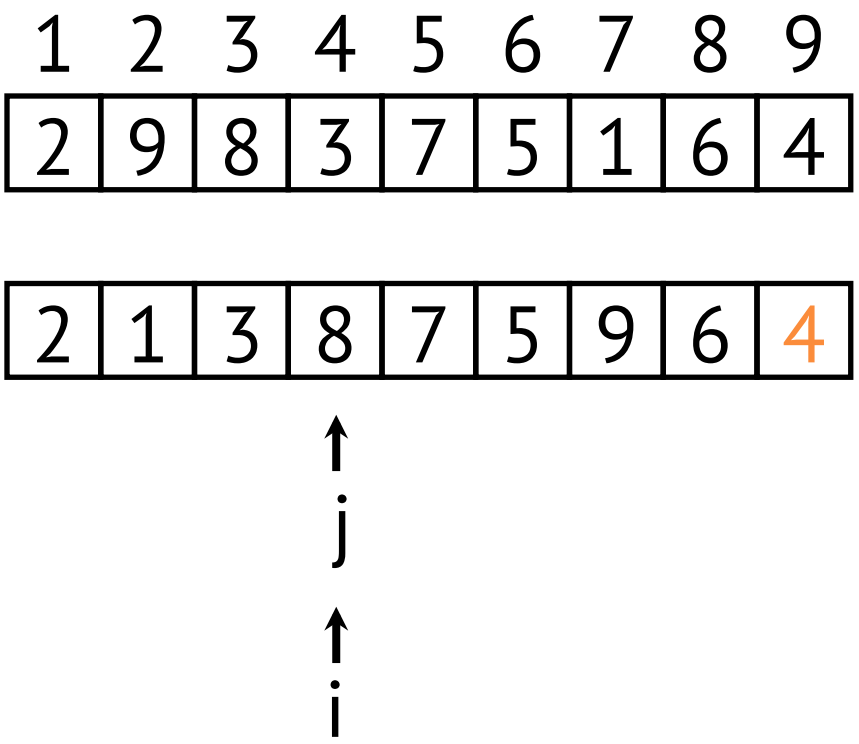

Beispiel

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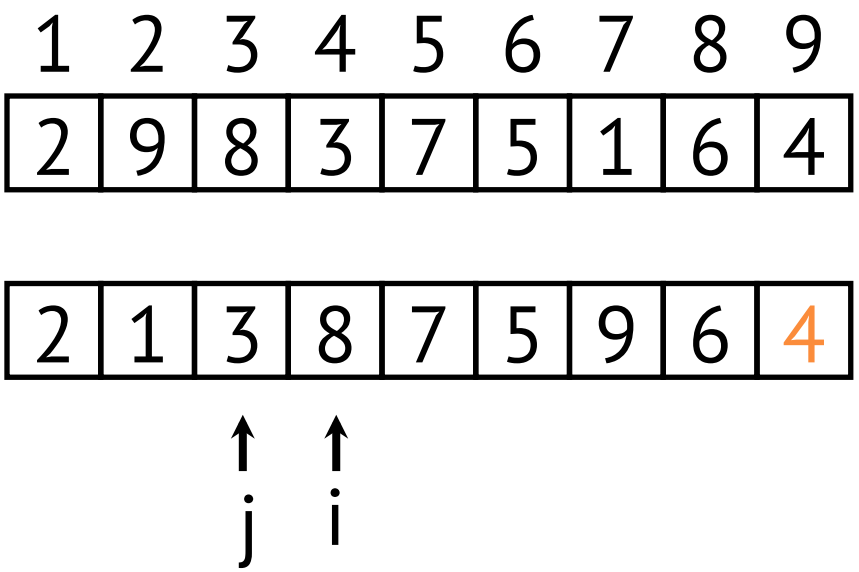

Beispiel

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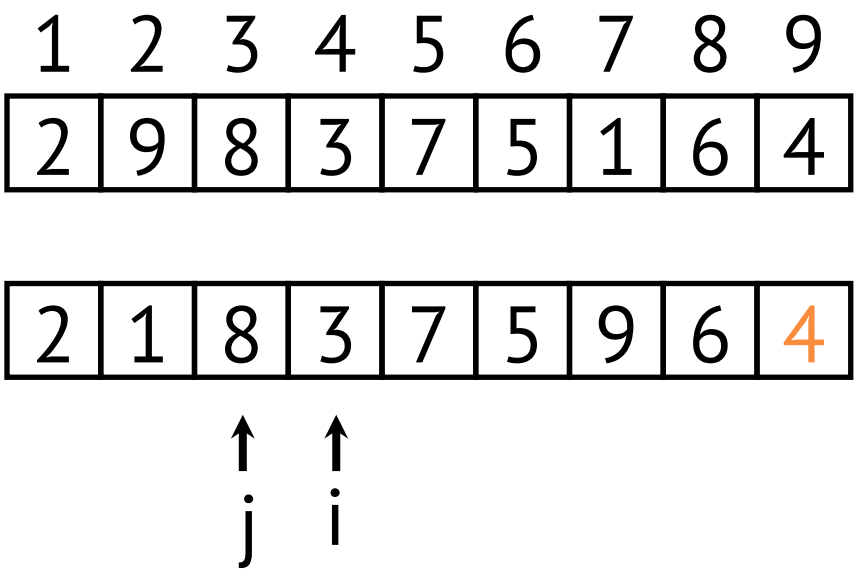

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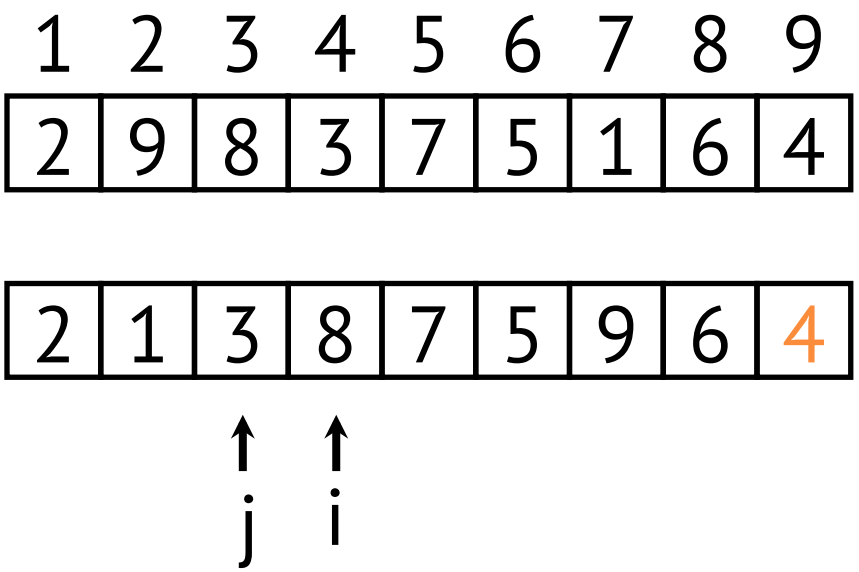

Beispiel

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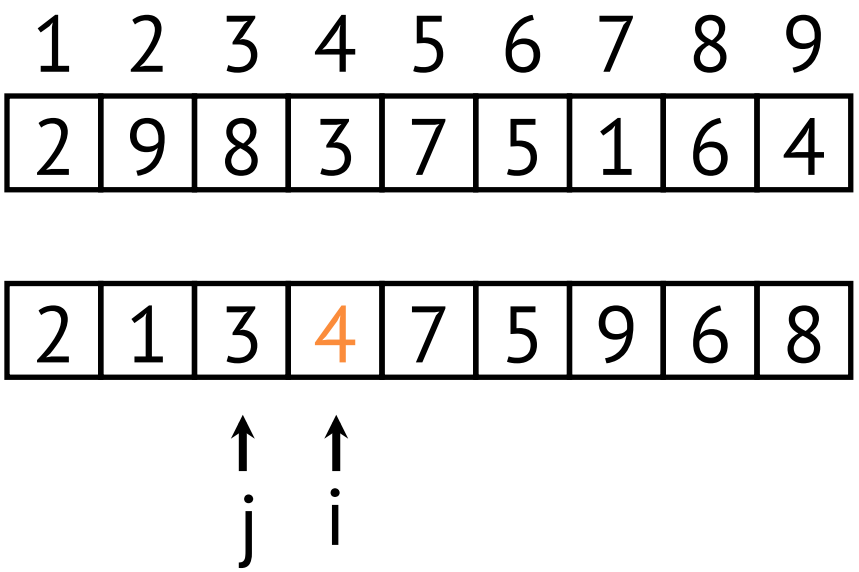


Beispiel

k = 6

1	2	3	4	5	6	7	8	9
2	9	8	3	7	5	1	6	4

2	1	3	4	7	5	9	6	8

m = 4, i = m - l + 1 = 4

Beispiel

k = 6

1	2	3	4	5	6	7	8	9
2	9	8	3	7	5	1	6	4

2	1	3	4	7	5	9	6	8

Beispiel

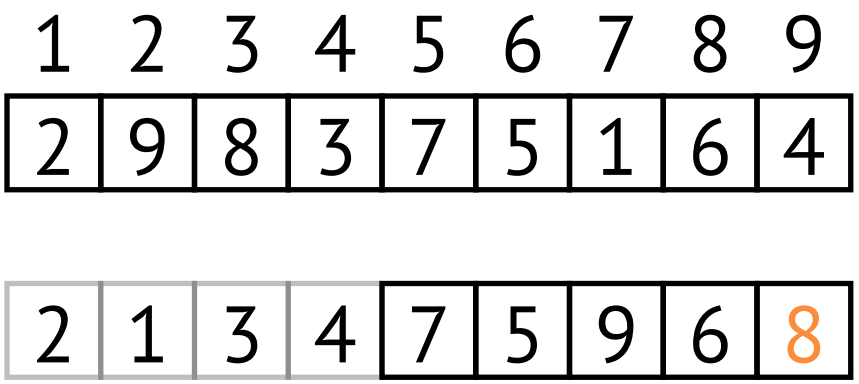
1	2	3	4	5	6	7	8	9
2	9	8	3	7	5	1	6	4

k = 6

2	1	3	4	7	5	9	6	8

k = 2

Beispiel



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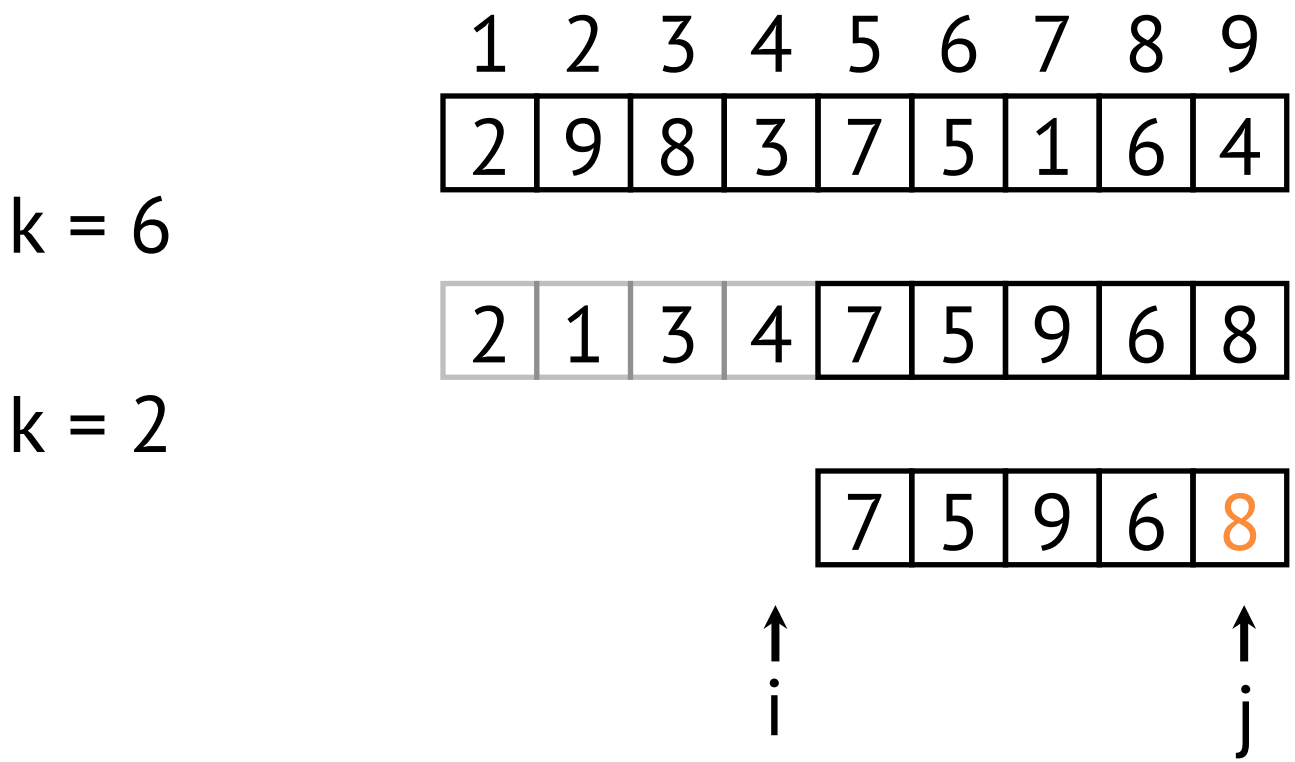

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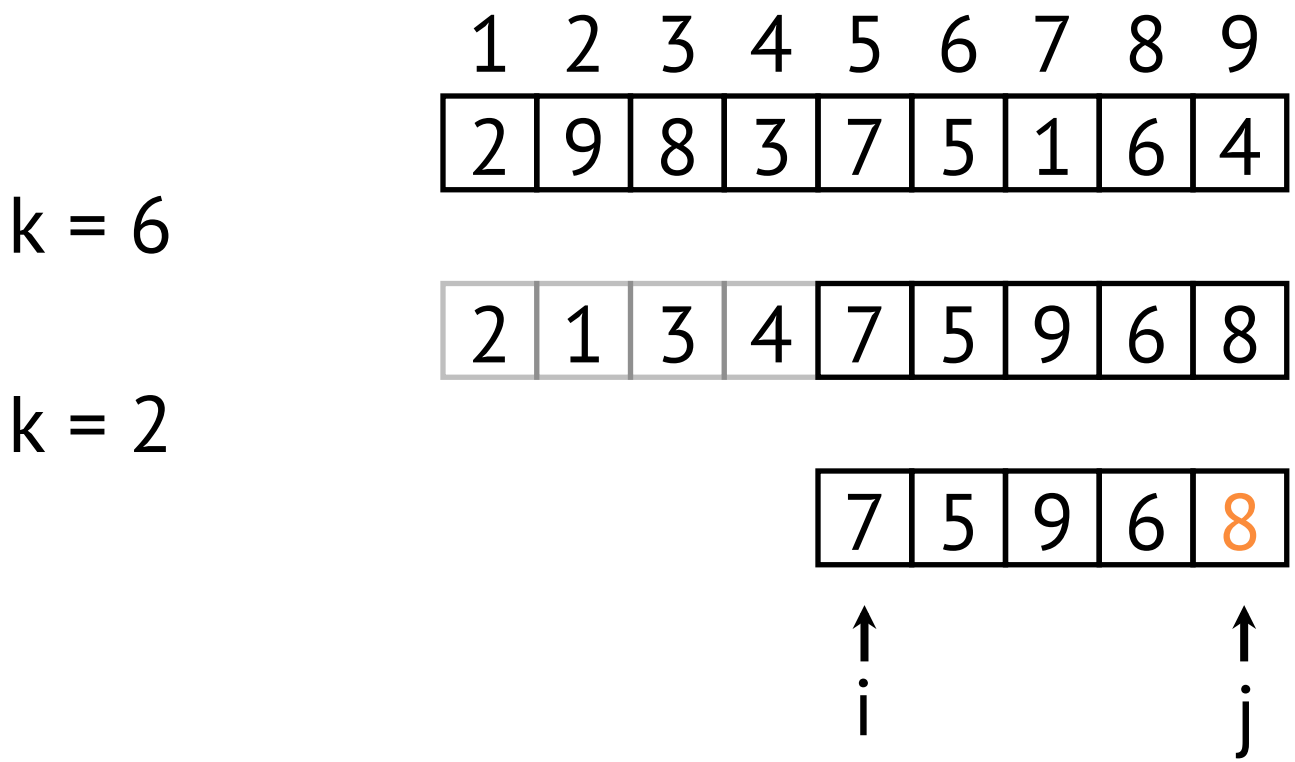

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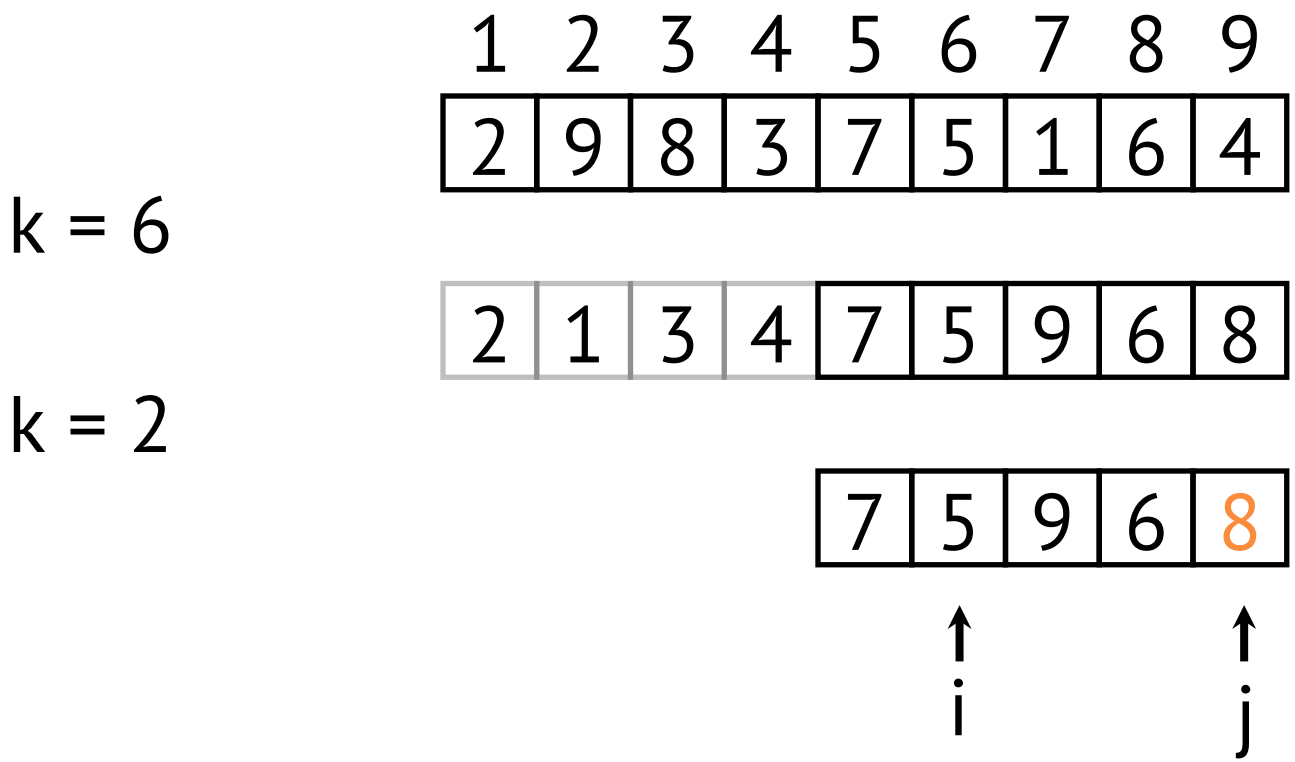

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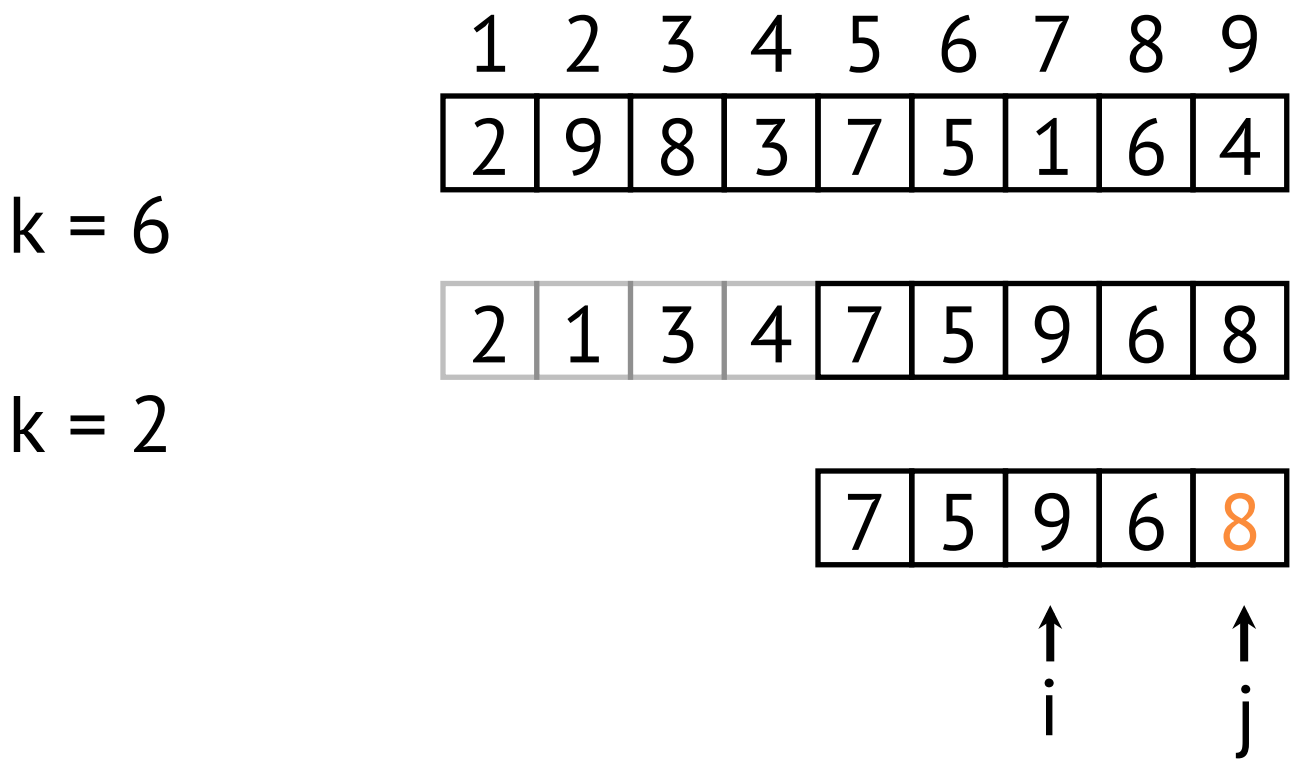

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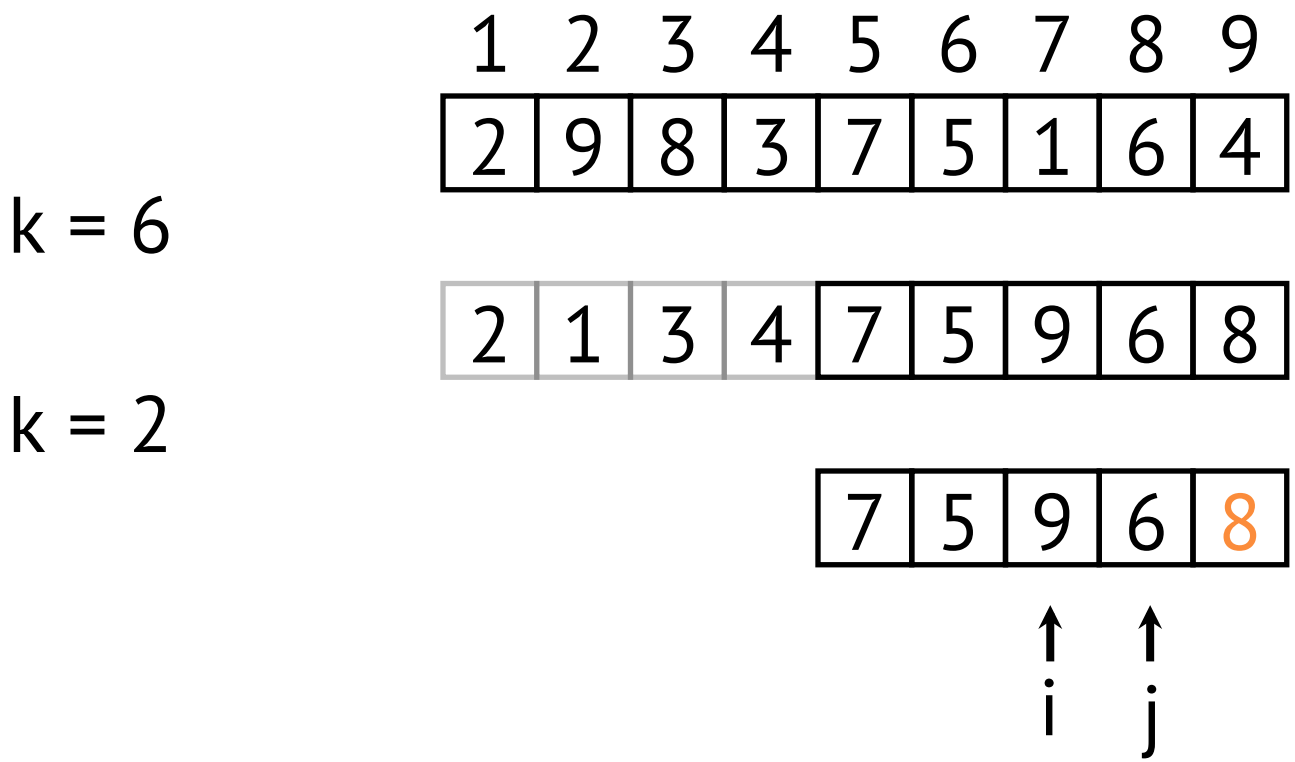

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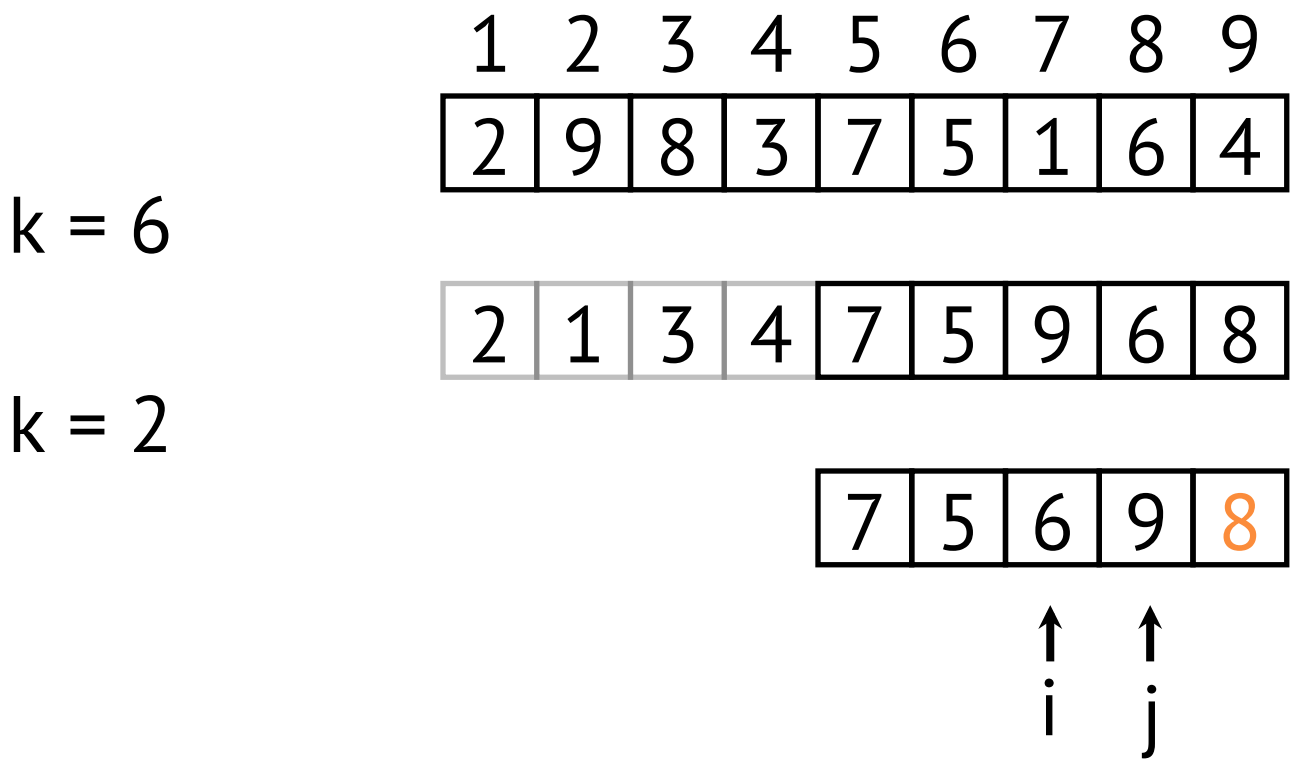

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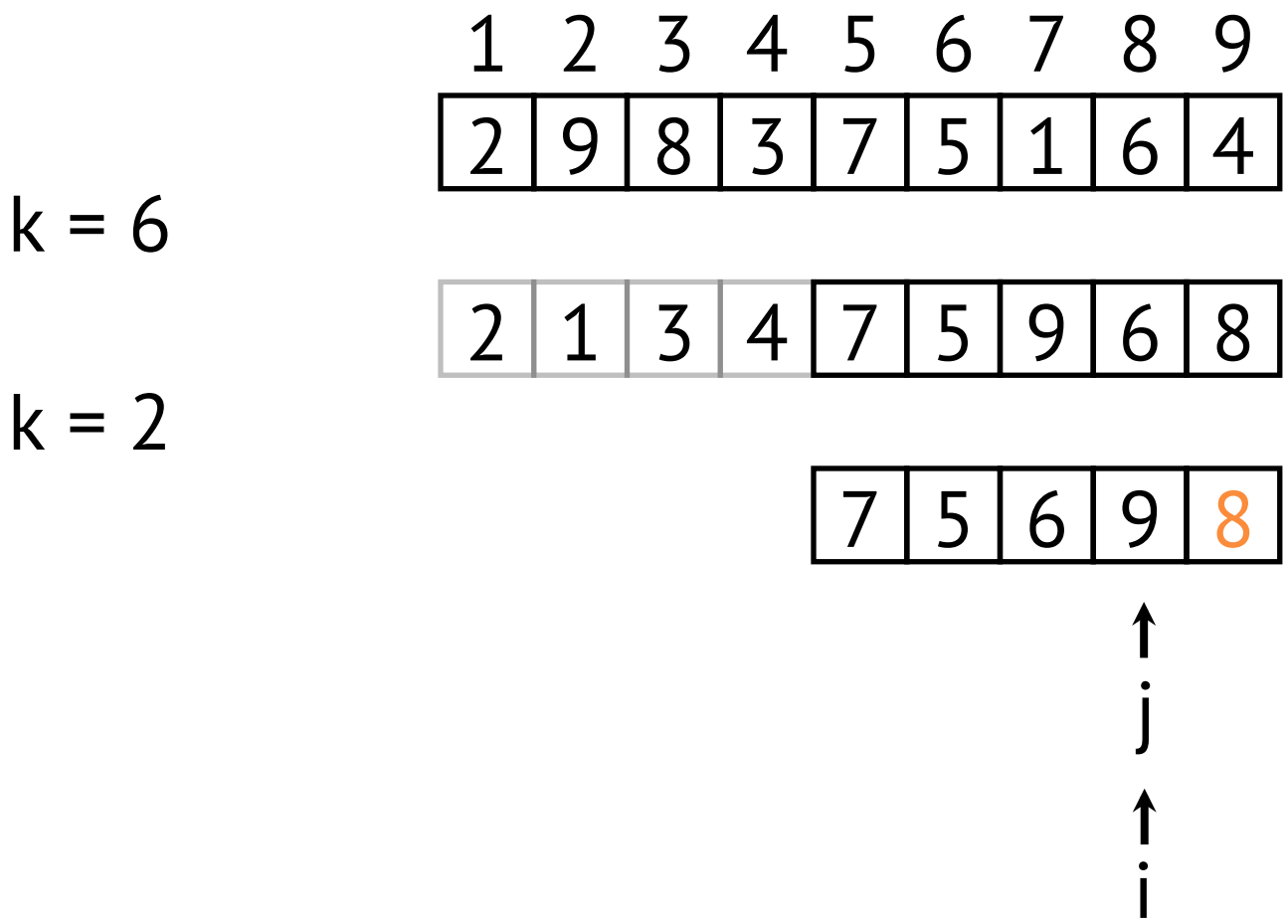

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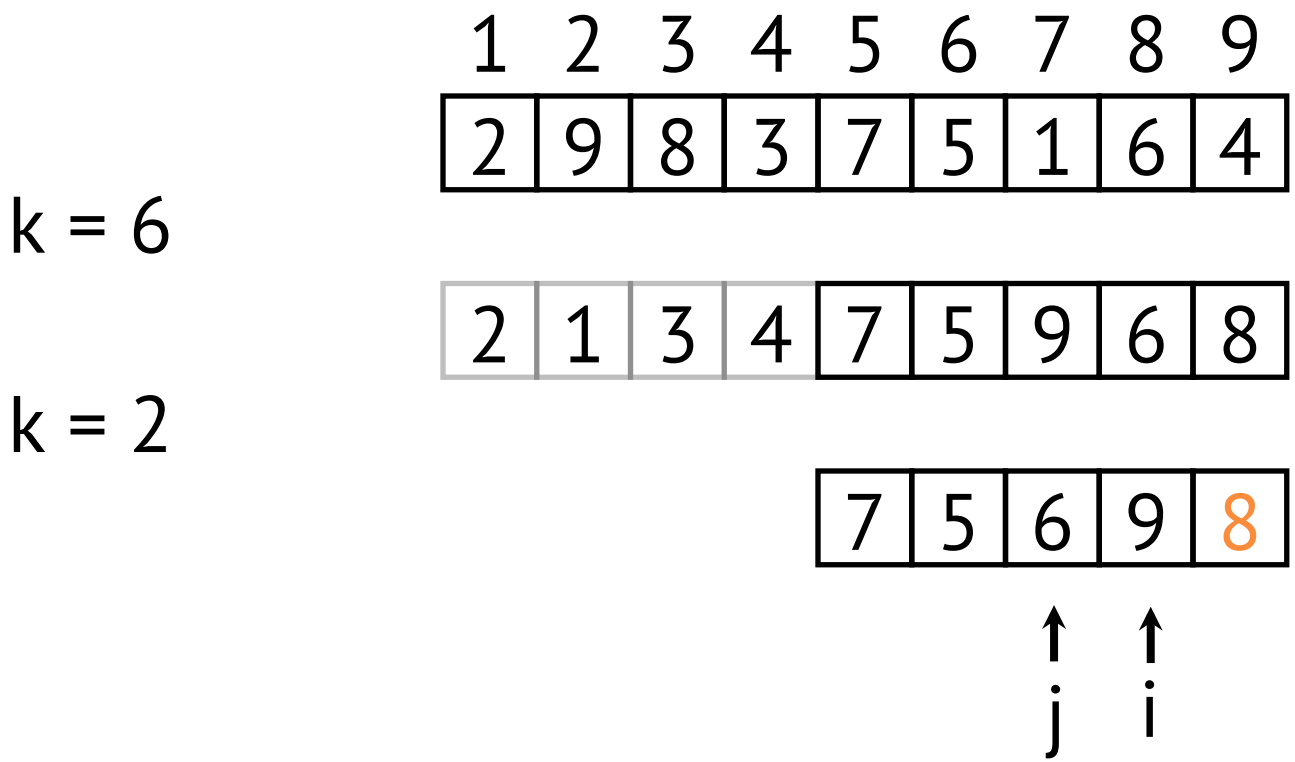

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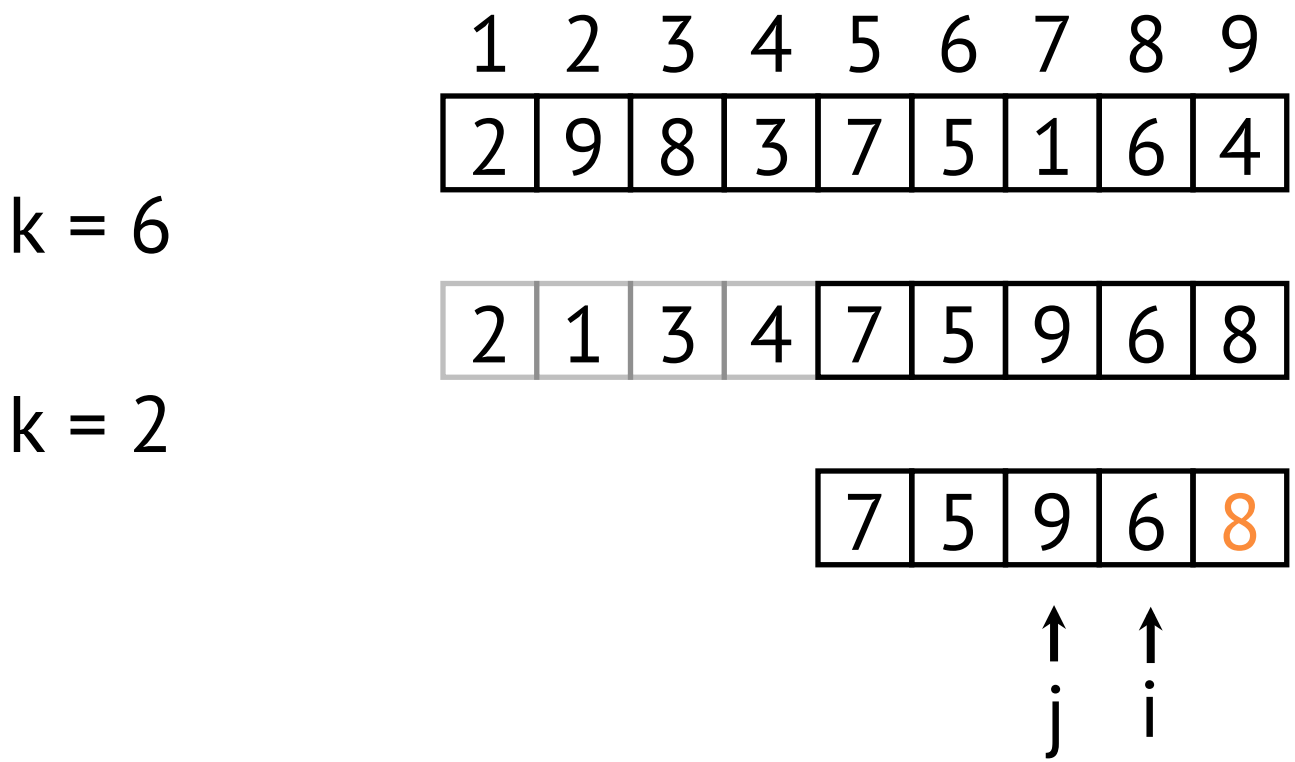

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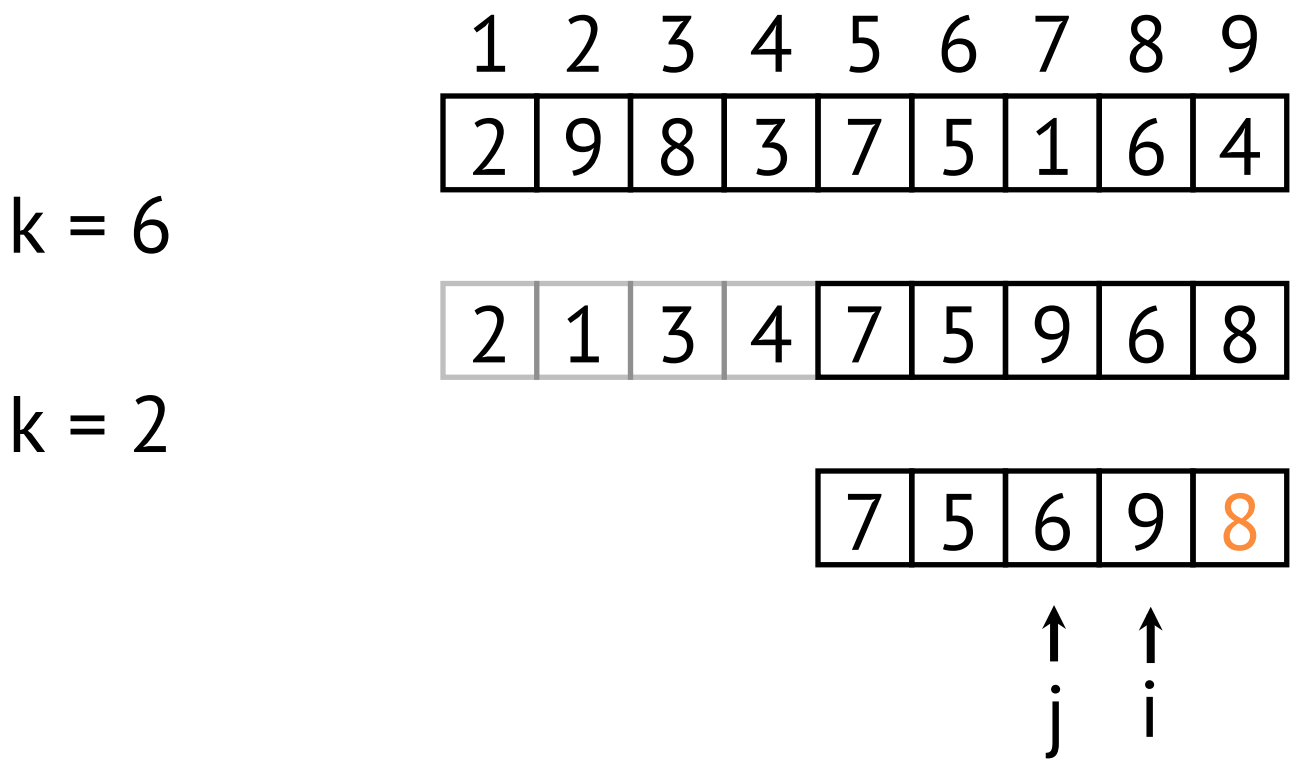
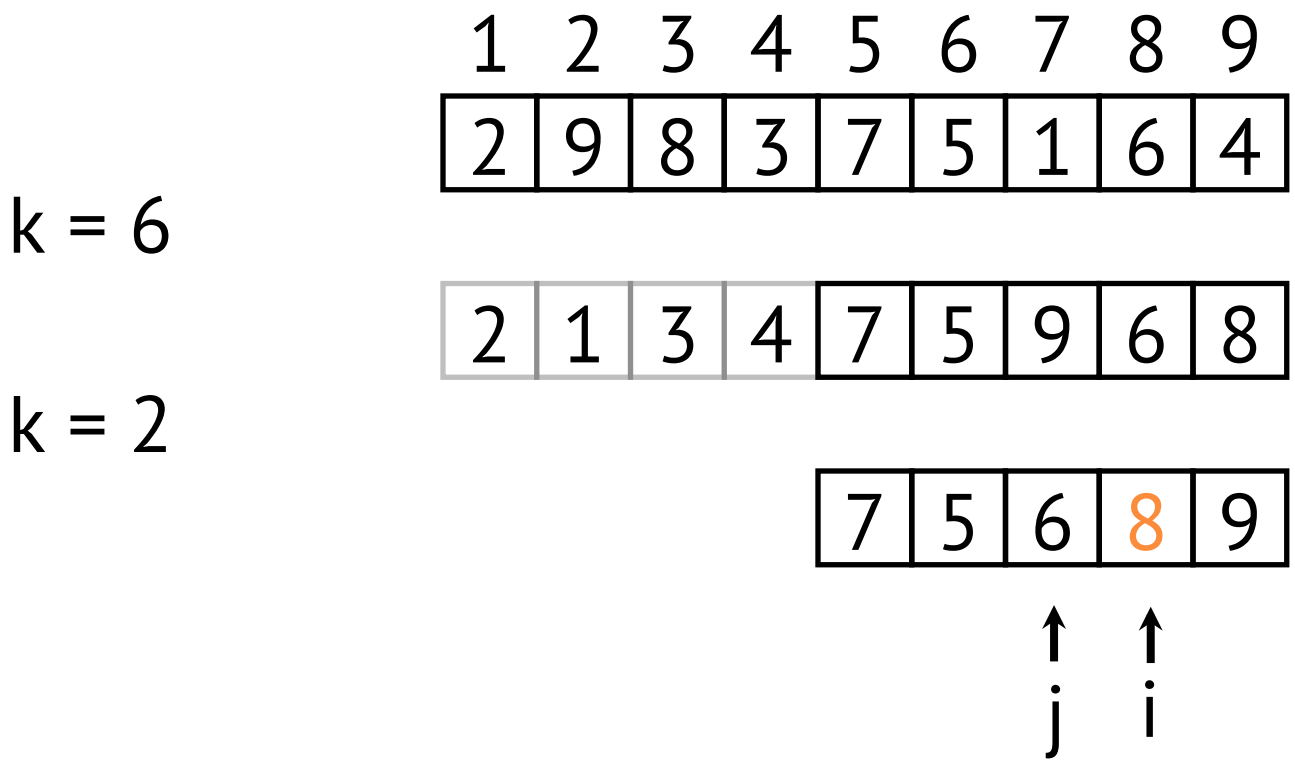


Beispiel



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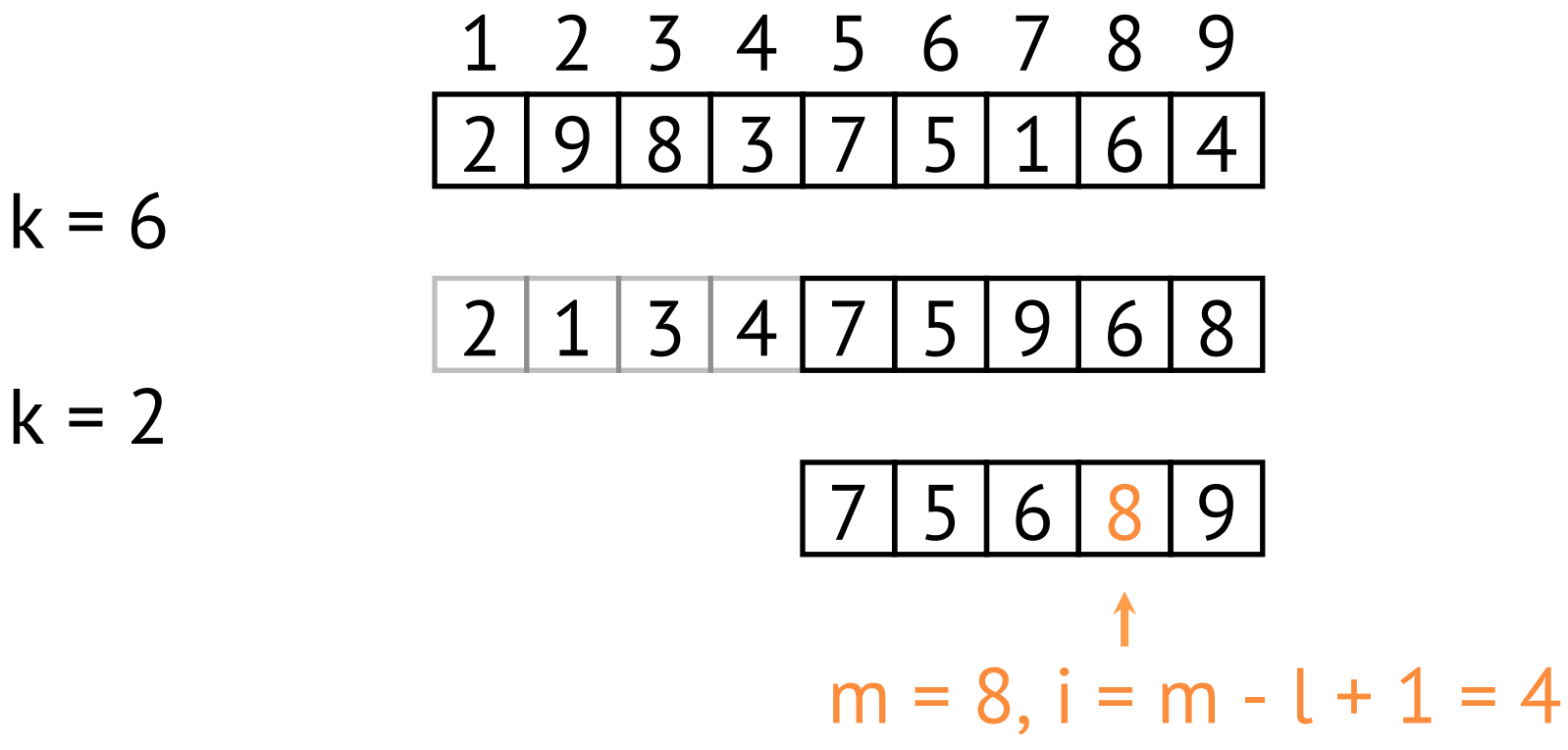
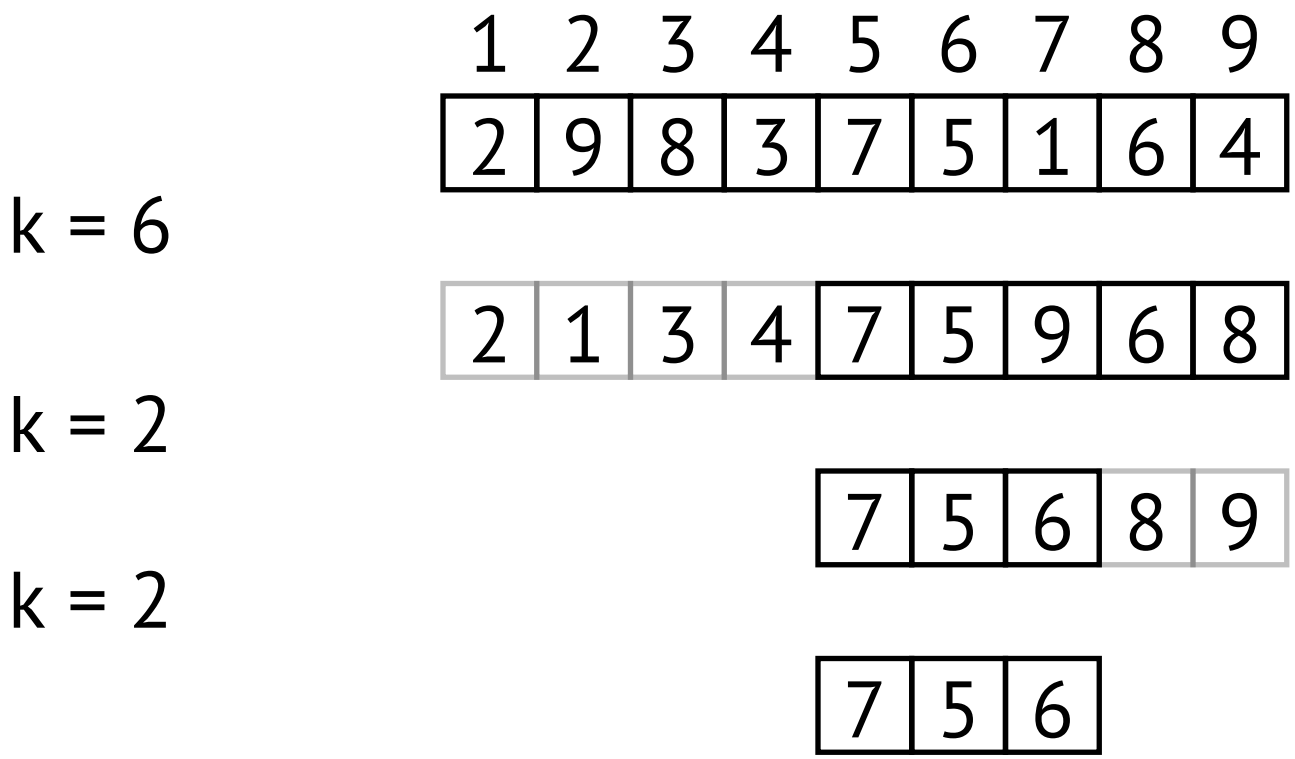


Beispiel



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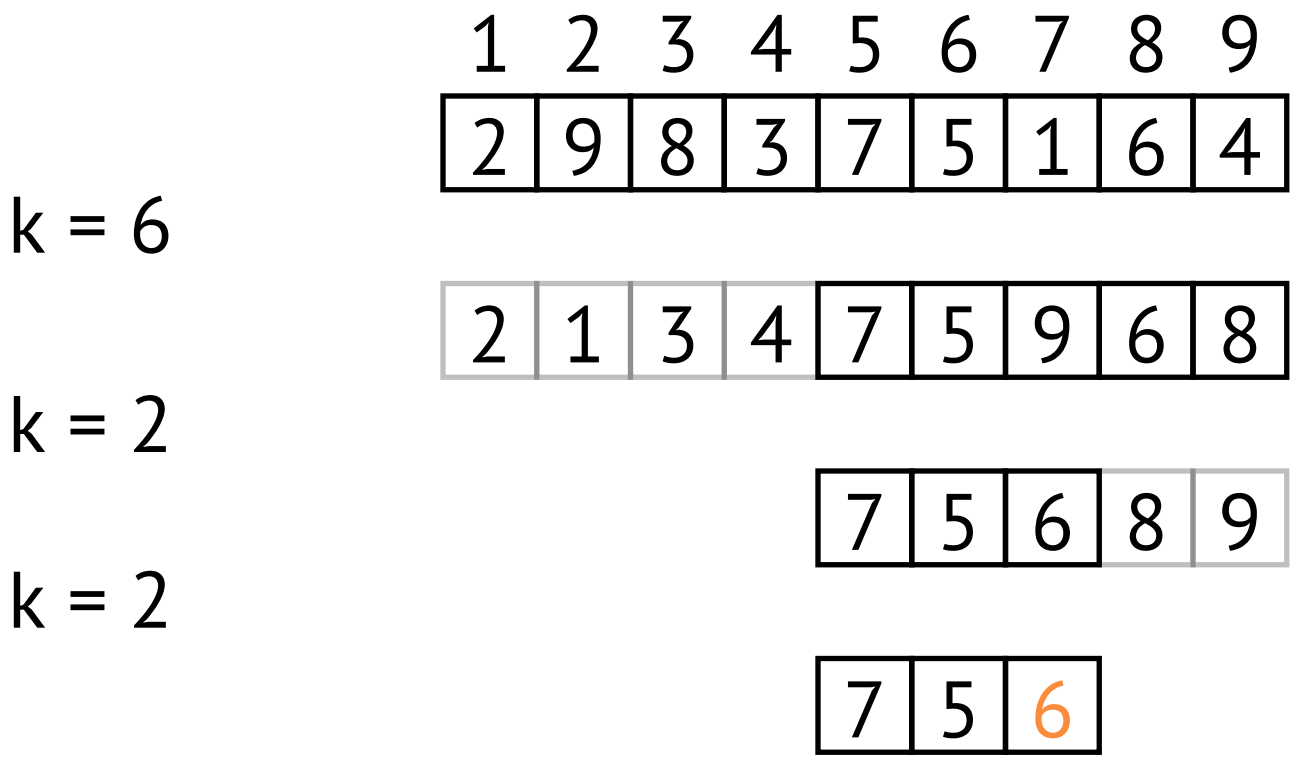
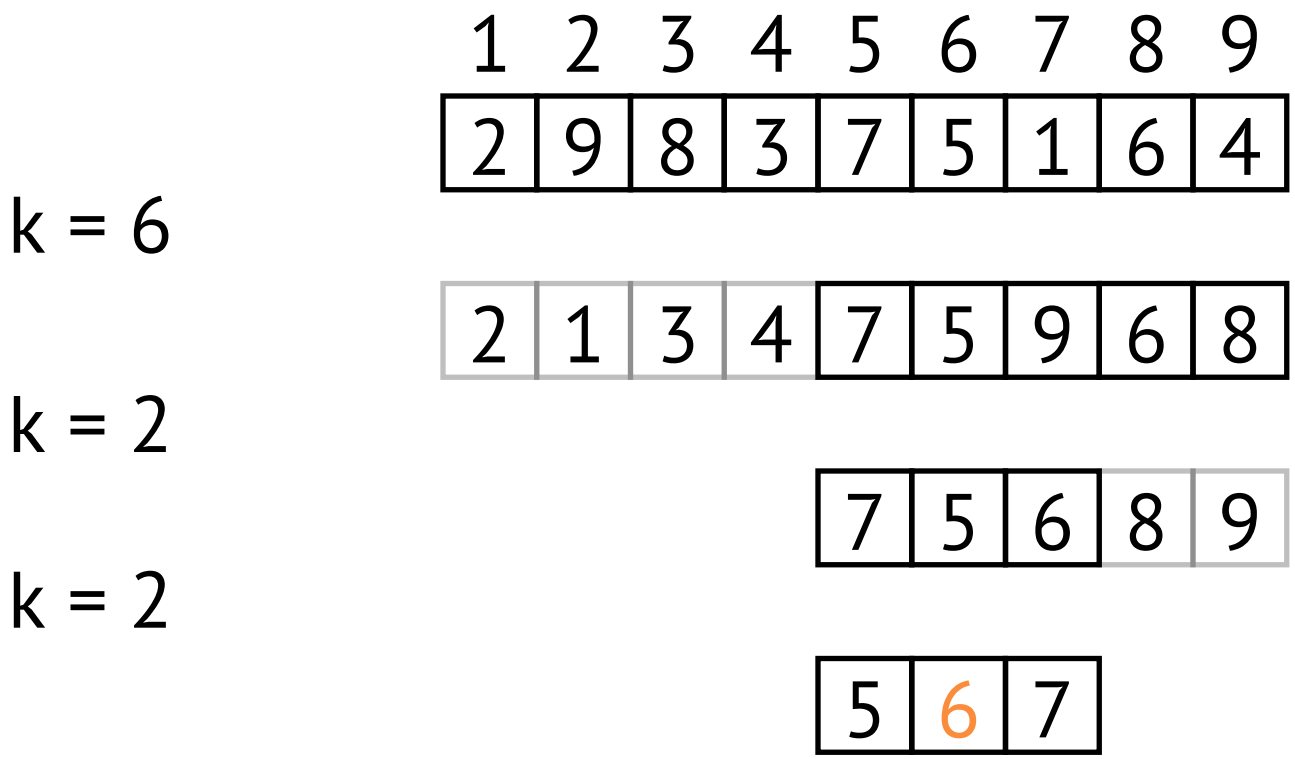


Beispiel



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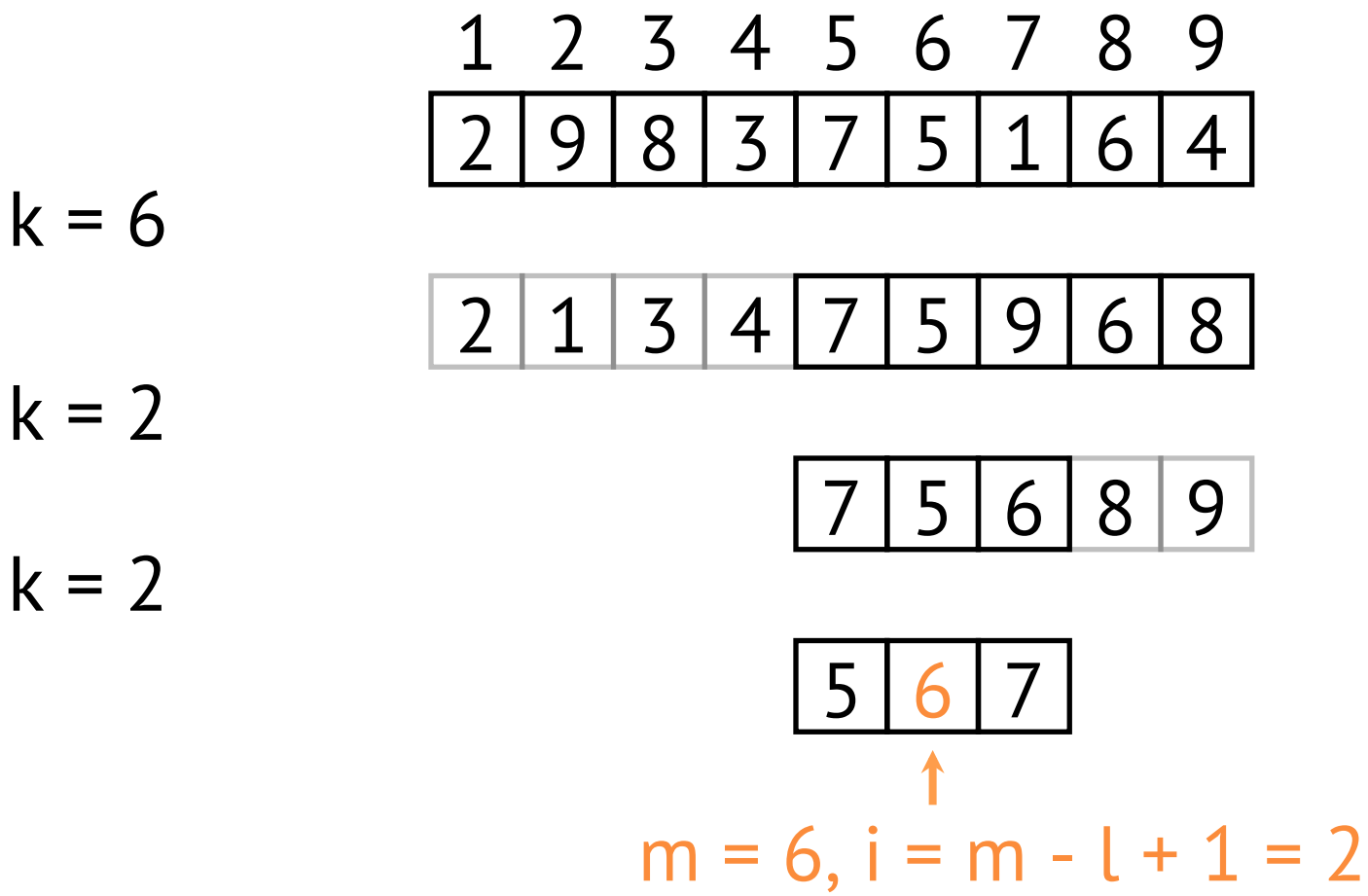
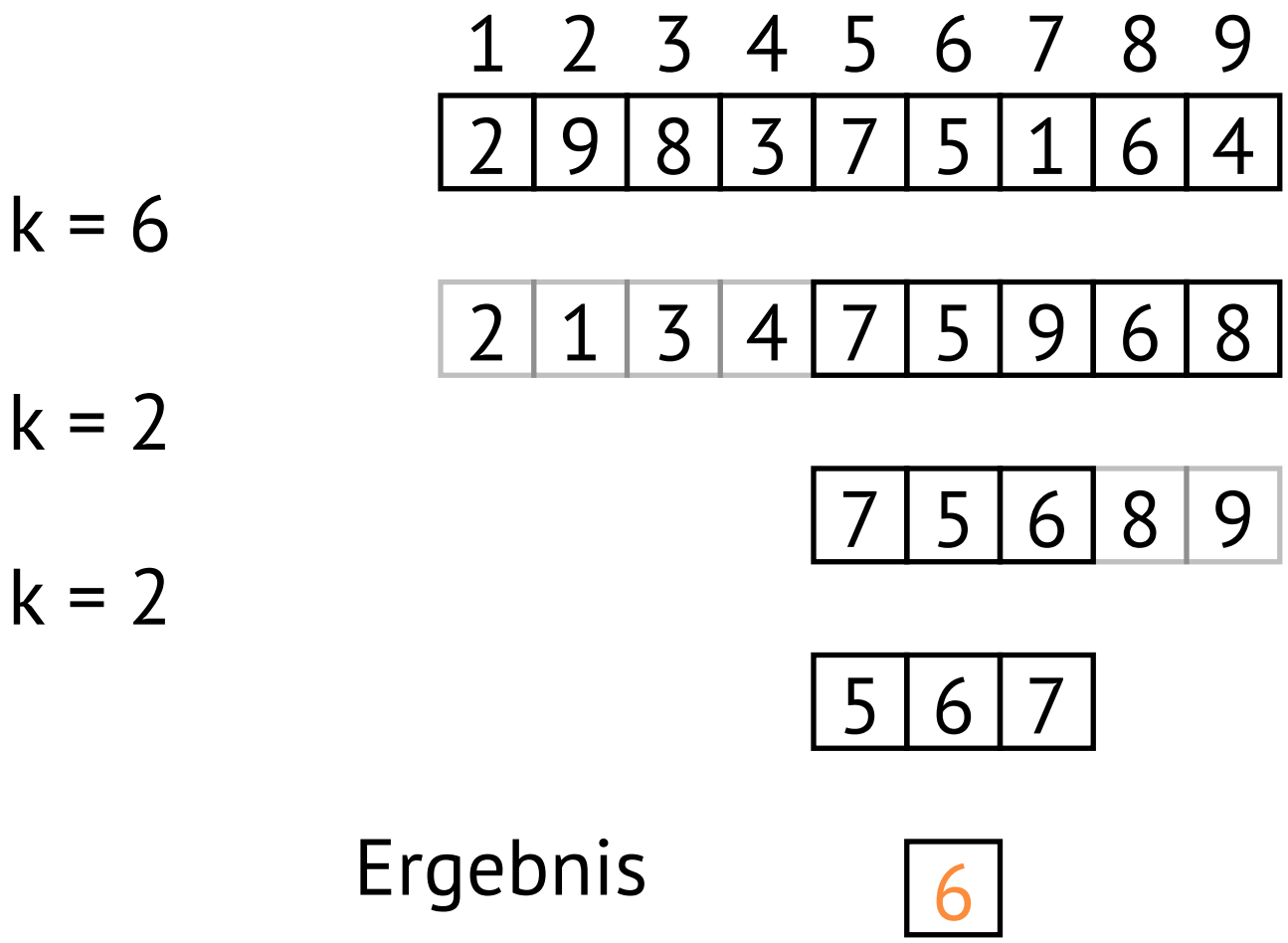


Beispiel


1 2 3 4 5 6 7 8 9
2 9 8 3 7 5 1 6 4
k = 6
2 1 3 4 7 5 9 6 8
k = 2
7 5 6 8 9
k = 2
5 6 7
Ergebnis
6

Median der Mediane

- Select(A, k)
  n ← |A|
  if n < 50 then
    return SortSelect(A, k)
  M ← { SortSelect(A[5i+1]...A[5i+5], 3) :
    i = 0,...,⌊n/5⌋-1}
  m ← Select(M, ⌊|M| / 2⌋)
  $S_<$ ← { A[i] : A[i] < m }
  $S_=$ ← { A[i] : A[i] = m }
  $S_>$ ← { A[i] : A[i] > m }
  if $|S_<| \geq k$ then
    return Select($S_<$, k)
  if $|S_<| + |S_=| = k$ then
    return m
  return Select($S_>$, k - $|S_<|$ - $|S_=|$)

Median der Mediane

- Select(A, k)
  - n ← |A|
  - if n < 50 then
    - return SortSelect(A, k)
  - M ← { SortSelect(A[5i+1]...A[5i+5], 3) : i = 0,...,⌊n/5⌋-1}
  - m ← Select(M, ⌊|M| / 2⌋)
  - $S_<$ ← { A[i] : A[i] < m }
  - $S_=$ ← { A[i] : A[i] = m }
  - $S_>$ ← { A[i] : A[i] > m }
  - if $|S_<| \geq k$ then
    - return Select($S_<$, k)
  - if $|S_<| + |S_=| = k$ then
    - return m
  - return Select($S_>$, k - $|S_<|$ - $|S_=|$)

Basisfall der Rekursion, die magische Konstante wird später klar.

Median der Mediane

- Select(A, k)
  - $n \leftarrow |A|$
  - if $n < 50$ then
    - return SortSelect(A, k)
  - $M \leftarrow \{ \text{SortSelect}(A[5i+1]...A[5i+5], 3) : i = 0,...,\lfloor n/5 \rfloor - 1 \}$
  - $m \leftarrow \text{Select}(M, \lfloor |M| / 2 \rfloor)$
  - $S_< \leftarrow \{ A[i] : A[i] < m \}$
  - $S_= \leftarrow \{ A[i] : A[i] = m \}$
  - $S_> \leftarrow \{ A[i] : A[i] > m \}$
  - if $|S_<| \geq k$ then
    - return Select($S_<$, k)
  - if $|S_<| + |S_=| = k$ then
    - return m
  - return Select($S_>$, $k - |S_<| - |S_=|$)

Bilde $\lfloor n/5 \rfloor$ Gruppen von je 5 Elementen (es bleiben 0-4 Elemente übrig) und bestimme deren Median.

Median der Mediane

- Select(A, k)
  - $n \leftarrow |A|$
  - if $n < 50$ then
    - return SortSelect(A, k)
  - $M \leftarrow \{$ SortSelect($A[5i+1]...A[5i+5]$, 3) : $i = 0,...,\lfloor n/5 \rfloor - 1\}$
  - $m \leftarrow$ Select(M, $\lfloor |M| / 2 \rfloor$)
  - $S_< \leftarrow \{ A[i] : A[i] < m \}$
  - $S_= \leftarrow \{ A[i] : A[i] = m \}$
  - $S_> \leftarrow \{ A[i] : A[i] > m \}$
  - if $|S_<| \geqslant k$ then
    - return Select($S_<$, k)
  - if $|S_<| + |S_=| = k$ then
    - return m
  - return Select($S_>$, $k - |S_<| - |S_=|$)

Bestimme rekursiv den Median der Mediane.

Median der Mediane

- Select(A, k)
  - $n \leftarrow |A|$
  - if $n < 50$ then
    - return SortSelect(A, k)
  - $M \leftarrow \{ \text{SortSelect}(A[5i+1]...A[5i+5], 3) : i = 0,...,\lfloor n/5 \rfloor - 1 \}$
  - $m \leftarrow \text{Select}(M, \lfloor |M| / 2 \rfloor)$
  - $S_< \leftarrow \{ A[i] : A[i] < m \}$
  - $S_= \leftarrow \{ A[i] : A[i] = m \}$
  - $S_> \leftarrow \{ A[i] : A[i] > m \}$
  - if $|S_<| \geqslant k$ then
    - return Select($S_<$, k)
  - if $|S_<| + |S_=| = k$ then
    - return m
  - return Select($S_>$, $k - |S_<| - |S_=|$)

Nach Voraussetzung sind alle Elemente verschieden und damit $|S_=| = |\{m\}| = 1$.

Median der Mediane

- Select(A, k)

```
n ← |A|
if n < 50 then
    return SortSelect( A, k )
M ← { SortSelect( A[5i+1]...A[5i+5], 3 ) :
        i = 0,...,⌊n/5⌋-1}
m ← Select( M, ⌊|M| / 2⌋)
S< ← { A[ i ] : A[ i ] < m }
S= ← { A[ i ] : A[ i ] = m }
S> ← { A[ i ] : A[ i ] > m }
if |S<| ⩾ k then
    return Select( S<, k )
if |S<| + |S=| = k then
    return m
return Select( S>, k - |S<| - |S=| )
```

Das k-te Element liegt in $S_<$

Median der Mediane

- Select(A, k)
  n ← |A|
  if n < 50 then
    return SortSelect(A, k)
  M ← { SortSelect(A[5i+1]...A[5i+5], 3) :
    i = 0,...,⌊n/5⌋-1}
  m ← Select(M, ⌊|M| / 2⌋)
  $S_<$ ← { A[i] : A[i] < m }
  $S_=$ ← { A[i] : A[i] = m }
  $S_>$ ← { A[i] : A[i] > m }
  if $|S_<| \geq k$ then
    return Select($S_<$, k)
  if $|S_<| + |S_=| = k$ then
    return m
  return Select($S_>$, k - $|S_<|$ - $|S_=|$)

Das k-te Element liegt in $S_==\{m\}$

Median der Mediane

- Select(A, k)
  - $n \leftarrow |A|$
  - if $n < 50$ then
    - return SortSelect(A, k)
  - $M \leftarrow \{ \text{SortSelect}(A[5i+1]...A[5i+5], 3) : i = 0,...,\lfloor n/5 \rfloor - 1 \}$
  - $m \leftarrow \text{Select}(M, \lfloor |M| / 2 \rfloor)$
  - $S_< \leftarrow \{ A[i] : A[i] < m \}$
  - $S_= \leftarrow \{ A[i] : A[i] = m \}$
  - $S_> \leftarrow \{ A[i] : A[i] > m \}$
  - if $|S_<| \geqslant k$ then
    - return Select($S_<$, k)
  - if $|S_<| + |S_=| = k$ then
    - return m
  - return Select($S_>$, $k - |S_<| - |S_=|$)

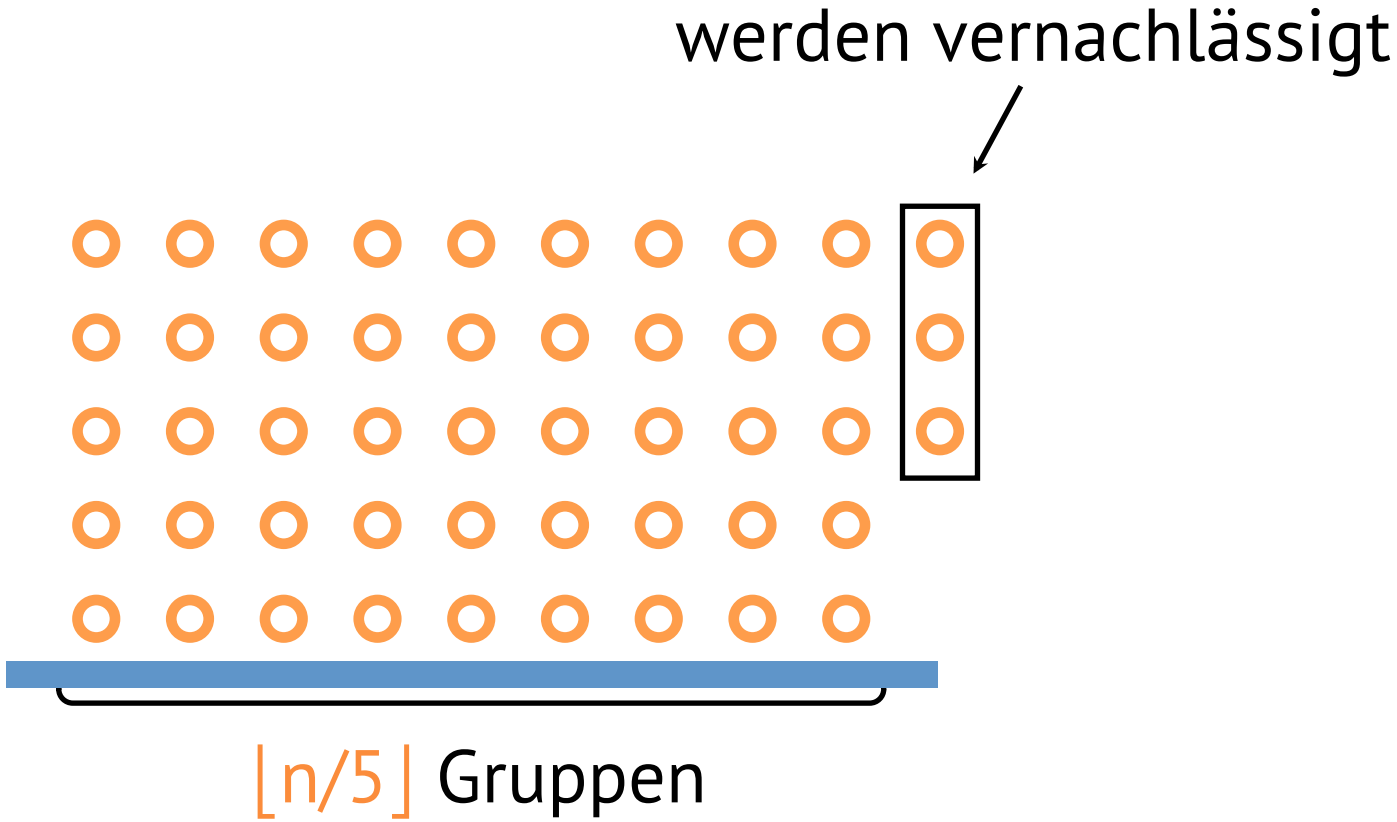
Das k-te Element ist das $(k-|S_<|-|S_=|)$ te Element aus $S_>$

Median der Mediane: Aufwand

Ausgangs-Situation:

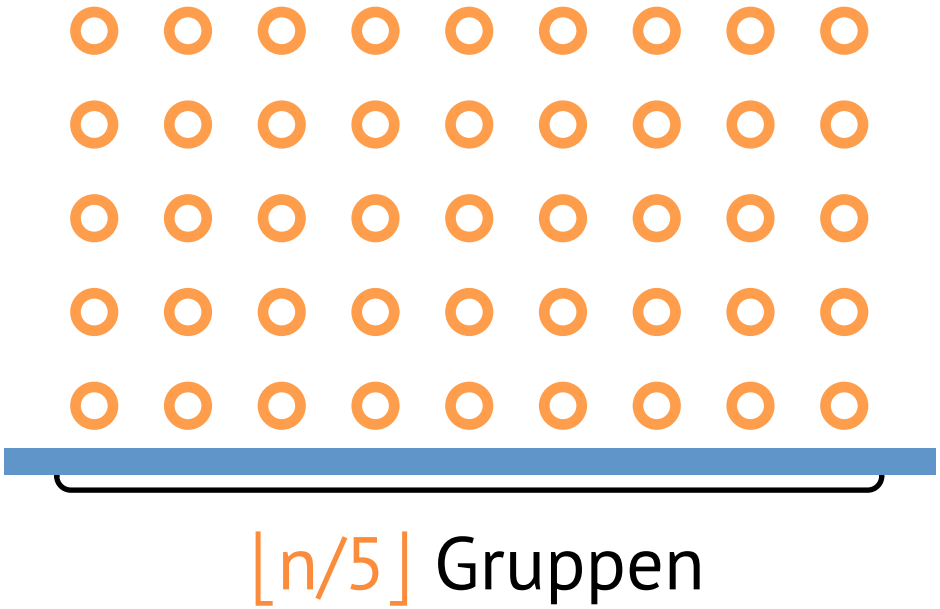
Median der Mediane: Aufwand

Anordnung in 5er Gruppen:



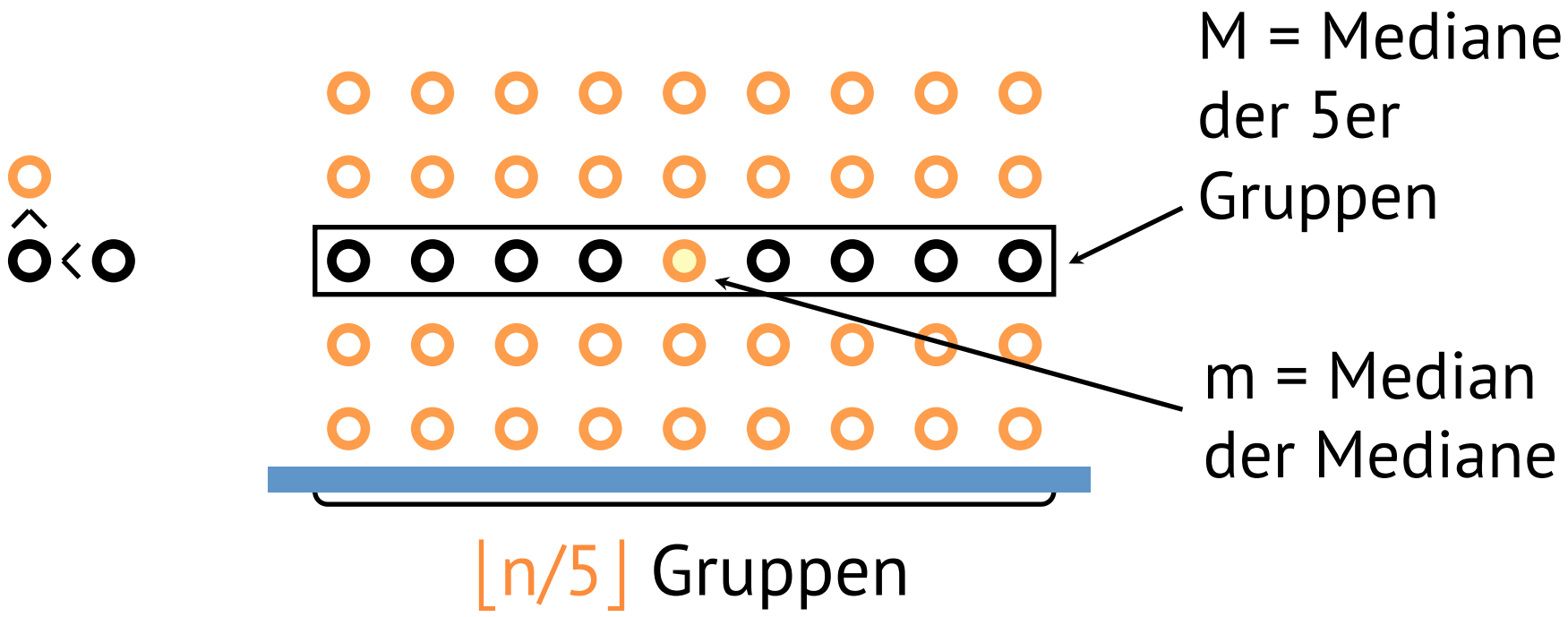
Median der Mediane: Aufwand

Anordnung in 5er Gruppen:



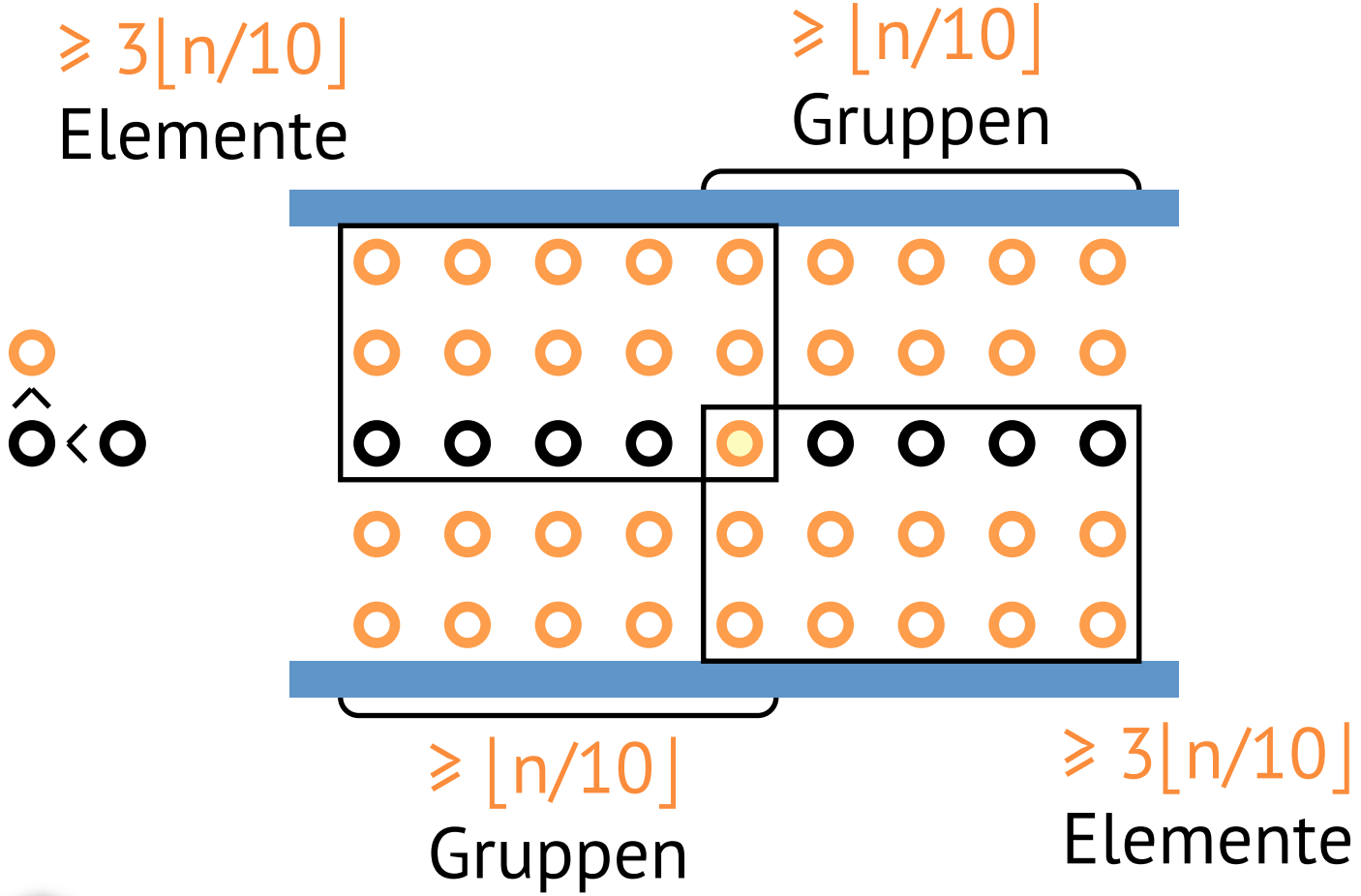
Median der Mediane: Aufwand

Anordnung in 5er Gruppen, sortiert dargestellt:



Median der Mediane: Aufwand

Anordnung in 5er Gruppen, sortiert dargestellt:



Median der Mediane: Aufwand

- mindestens $3\lfloor n/10 \rfloor$ Elemente sind $\leqslant m$
- höchstens $n - 3\lfloor n/10 \rfloor$ Elemente sind $> m$
- für $n > 50$ ist $|S_{>}| < n - 3\lfloor n/10 \rfloor < 3n/4$
- für $n > 50$ ist $|S_{<}| < n - 3\lfloor n/10 \rfloor < 3n/4$

n=47	$n - 3\lfloor n/10 \rfloor = 35$	$3n/4 = 35.25$
n=48	$n - 3\lfloor n/10 \rfloor = 36$	$3n/4 = 36.00$
n=49	$n - 3\lfloor n/10 \rfloor = 37$	$3n/4 = 36.75$
n=50	$n - 3\lfloor n/10 \rfloor = 35$	$3n/4 = 37.50$
n=51	$n - 3\lfloor n/10 \rfloor = 36$	$3n/4 = 38.25$

Median der Aufwand

- Select(A, k)

```
n ← |A|
if n < 50 then                                              O(1)
    return SortSelect( A, k )
M ← { SortSelect( A[5i+1]...A[5i+5], 3 ) :                  O(n)
        i = 0,...,⌊n/5⌋-1}
m ← Select( M, ⌊|M| / 2⌋)                                   T(n/5)
S< ← { A[ i ] : A[ i ] < m }                                O(n)
S= ← { A[ i ] : A[ i ] = m }
S> ← { A[ i ] : A[ i ] > m }
if |S<| ⩾ k then                                            <T(3n/4)
    return Select( S<, k )
if |S<| + |S=| = k then
    return m
return Select( S>, k - |S<| - |S=| )
```

Median der Aufwand

- Rekurrenz

$$T(n) \leq \begin{cases} c_1 & \text{für} \quad n < 50 \\ c_2 n + T(\frac{n}{5}) + T(\frac{3n}{4}) & \text{für} \quad n \geq 50 \end{cases}$$

- Behauptung: für $c=\max(c_1, 20c_2)$ ist

$$T(n) < cn = O(n)$$

Median der Aufwand

- Induktionsanfang: für $n<50$ okay, da $c>c_1$
- Induktionsannahme: $T(m) \leqslant cm$ für $m<n$
- Induktionsschluss:

$$\begin{aligned} T(n) &\leq c_2 n + T(n/5) + T(3n/4) \\ &\leq c_2 n + cn/5 + 3cn/4 \\ &\leq c_2 n + 19cn/20 \\ &\leq cn + n(c_2 - c/20) \\ &\leq cn \end{aligned}$$

$$20c_2 \leq c \Leftrightarrow c_2 \leq c/20 \Leftrightarrow c_2 - c/20 \leq 0$$

Selektion: Zusammenfassung

- Selektion durch Sortieren: $O(n \times \log n)$
- Selektion durch Partitionieren
  - Worst case: $O(n^2)$
  - Average case: $O(n)$
- Median der Mediane: $O(n)$